First Midterm Exam - May 28, 2025, Stochastic Analysis

Let's tackle these two stochastic analysis problems.

Exercise 1: Rescaling an Itô Integral to be a Brownian Motion

Let (B_t) denote standard Brownian motion. Let $X_t = \int_0^t e^{3s}\,\mathrm{d}B_s$. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ denote a monotone increasing continuous deterministic function satisfying $f(0) = 0$. We want to choose f such that the continuous-time process $(X_{f(t)})_{t\geq 0}$ has the same law as (B_t).

Hint: Use the equivalent characterization of Brownian motion as a Gaussian process with a given autocovariance structure.

A standard Brownian motion (B_t) is a Gaussian process with $E[B_t] = 0$ for all t, and its autocovariance function is $E[B_sB_t] = \min(s,t)$.

First, let's analyze the process X_t. X_t is an Itô integral. It is a continuous martingale with $X_0 = 0$. The quadratic variation of X_t is given by: $\langle X\rangle_t = \int_0^t (e^{3s})^2\,\mathrm{d}s = \int_0^t e^{6s}\,\mathrm{d}s$. Since X_t is a continuous martingale starting at 0, and its quadratic variation is deterministic, by Lévy's characterization of Brownian motion, X_t is a time-changed Brownian motion. Specifically, $X_t = B'_{\langle X\rangle_t}$ for some standard Brownian motion B'. Let $Y_t = X_{f(t)}$. We want Y_t to be a standard Brownian motion. This means Y_t must be a Gaussian process, $E[Y_t] = 0$, and $E[Y_sY_t] = \min(s,t)$.

Let's check the mean of X_t: $E[X_t] = E\left[\int_0^t e^{3s}\,\mathrm{d}B_s\right] = 0$ for all t. Therefore, $E[X_{f(t)}] = 0$ for all t. This condition is satisfied.

Now let's compute the autocovariance of X_t: For $s \leq t$, $E[X_sX_t] = E\left[\left(\int_0^s e^{3u}\,\mathrm{d}B_u\right)\left(\int_0^t e^{3v}\,\mathrm{d}B_v\right)\right]$. Using the Itô isometry for two Itô integrals, if $X_t = \int_0^t H_u dB_u$ and $Y_t = \int_0^t G_u dB_u$, then $E[X_tY_t] = E[\int_0^t H_uG_u du]$. Here, we have X_s and X_t. We can rewrite $X_t = X_s + \int_s^t e^{3v} dB_v$. $E[X_sX_t] = E\left[X_s\left(X_s + \int_s^t e^{3v}\,\mathrm{d}B_v\right)\right] = E[X_s^2] + E\left[X_s\int_s^t e^{3v}\,\mathrm{d}B_v\right]$. Since X_s is $\mathcal{F}_s$-measurable and $\int_s^t e^{3v}\,\mathrm{d}B_v$ is a martingale increment independent of $\mathcal{F}_s$ (because $B_v - B_s$ for $v > s$ is independent of $\mathcal{F}_s$), the second term is $E[X_s]E[\int_s^t e^{3v}\,\mathrm{d}B_v] = 0\cdot 0 = 0$. So, $E[X_sX_t] = E[X_s^2]$ for $s \leq t$. By Itô isometry, $E[X_s^2] = E\left[\left(\int_0^s e^{3u}\,\mathrm{d}B_u\right)^2\right] = \int_0^s (e^{3u})^2\,\mathrm{d}u = \int_0^s e^{6u}\,\mathrm{d}u$.

Therefore, for $s, t \geq 0$, $E[X_sX_t] = \int_0^{\min(s,t)} e^{6u}\,\mathrm{d}u$.

Now consider the process $Y_t = X_{f(t)}$. We want Y_t to have the same law as B_t. This means its autocovariance must be $E[Y_sY_t] = \min(s,t)$. $E[Y_sY_t] = E[X_{f(s)}X_{f(t)}]$. Let $s' = f(s)$ and $t' = f(t)$. $E[X_{s'}X_{t'}] = \int_0^{\min(s',t')} e^{6u}\,\mathrm{d}u$. Substituting back s' and t': $E[Y_sY_t] = \int_0^{\min(f(s),f(t))} e^{6u}\,\mathrm{d}u$.

We require this to be equal to $\min(s,t)$. So, $\int_0^{\min(f(s),f(t))} e^{6u}\,\mathrm{d}u = \min(s,t)$.

Let's evaluate the integral: $\int_0^x e^{6u}\,\mathrm{d}u = \left[\frac{1}{6}e^{6u}\right]_0^x = \frac{1}{6}(e^{6x} - e^0) = \frac{1}{6}(e^{6x} - 1)$.

So, we need $\frac{1}{6}(e^{6\min(f(s),f(t))} - 1) = \min(s,t)$. Since f is monotone increasing, $\min(f(s), f(t)) = f(\min(s,t))$. Let $u = \min(s,t)$. Then we need: $\frac{1}{6}(e^{6f(u)} - 1) = u$.

Now we need to solve for $f(u)$: $e^{6f(u)} - 1 = 6u$ $e^{6f(u)} = 6u + 1$ $6f(u) = \ln(6u+1)$ $f(u) = \frac{1}{6}\ln(6u+1)$.

Therefore, the function $f(t) = \frac{1}{6}\ln(6t+1)$ is the required choice.

Exercise 2: Brownian Motion with Upward Drift and Hitting Times

Let (B_t) denote the standard Brownian motion. Let $0 \leq \mu < +\infty$. Let $X_t := B_t + \mu t$. We call (X_t) the Brownian motion with upward drift μ. Let $\tau = \min\{\, t \ : \ X_t = x \,\}$ denote the hitting time of level $x \in \mathbb{R}_+$. You may assume without proof that $\mathbb{P}(\tau < +\infty) = 1$.

Part (a): For any $\alpha \in \mathbb{R}$ find a constant $\beta \in \mathbb{R}$ such that (M_t) is a martingale, where $M_t := \exp(\alpha X_t - \beta t)$.

Hint: $\exp\left(\lambda B_t - \frac{\lambda^2}{2}t\right)$ is a martingale for any choice of $\lambda \in \mathbb{R}$.

Let $M_t = \exp(\alpha X_t - \beta t)$. Substitute $X_t = B_t + \mu t$: $M_t = \exp(\alpha(B_t + \mu t) - \beta t)$ $M_t = \exp(\alpha B_t + \alpha\mu t - \beta t)$ $M_t = \exp(\alpha B_t + (\alpha\mu - \beta)t)$.

For M_t to be a martingale, comparing it to the given hint form $\exp\left(\lambda B_t - \frac{\lambda^2}{2}t\right)$, we need: The coefficient of B_t is α, so $\lambda = \alpha$. The coefficient of t must be $-\frac{\lambda^2}{2}$, which is $-\frac{\alpha^2}{2}$. So we need $(\alpha\mu - \beta) = -\frac{\alpha^2}{2}$.

Now we can solve for β: $\beta = \alpha\mu + \frac{\alpha^2}{2}$.

Thus $M_t = \exp\left(\alpha X_t - \left(\alpha\mu + \frac{\alpha^2}{2}\right)t\right)$ is a martingale.

Part (b): Apply the optional stopping theorem to calculate $\mathbb{E}\left(e^{-\lambda\tau}\right)$ for any $\lambda \in \mathbb{R}_+$.

We want to calculate $\mathbb{E}(e^{-\lambda\tau})$. Let's choose α and β in part (a) such that M_t is related to $e^{-\lambda t}$. From part (a), $M_t = \exp\left(\alpha X_t - \left(\alpha\mu + \frac{\alpha^2}{2}\right)t\right)$. We want the coefficient of t in the exponent to be $-\lambda$. So, we need $-\left(\alpha\mu + \frac{\alpha^2}{2}\right) = -\lambda$. $\alpha\mu + \frac{\alpha^2}{2} = \lambda$. $\frac{1}{2}\alpha^2 + \mu\alpha - \lambda = 0$.

This is a quadratic equation in α. We can solve for α using the quadratic formula: $\alpha = \frac{-\mu \pm \sqrt{\mu^2 - 4(\frac{1}{2})(-\lambda)}}{2(\frac{1}{2})}$ $\alpha = \frac{-\mu \pm \sqrt{\mu^2 + 2\lambda}}{1}$ $\alpha = -\mu \pm \sqrt{\mu^2 + 2\lambda}$.

Since $\lambda > 0$, $\mu^2 + 2\lambda > \mu^2$, so $\sqrt{\mu^2 + 2\lambda} > \mu$. This means one root for α is positive $(-\mu + \sqrt{\mu^2 + 2\lambda})$, and the other is negative $(-\mu - \sqrt{\mu^2 + 2\lambda})$.

Let's define $M_t = \exp(\alpha X_t - \lambda t)$ where α is one of the roots. Since τ is a hitting time, $X_\tau = x$ (assuming $X_0 = 0$, which is the usual start for Brownian motion and its drift, but the question doesn't specify X_0. If $X_0 = 0$, then $X_\tau = x$. If $X_0 = x_0$, then $X_\tau = x$. Let's assume $X_0 = 0$ for now, then τ is the first time X_t reaches x, starting from 0). Then $M_\tau = \exp(\alpha x - \lambda\tau)$.

We are given the condition for Optional Stopping Theorem: if (M_t) is a martingale, τ is a stopping time satisfying $\mathbb{P}(\tau < +\infty) = 1$ and if there exists a constant $C \in \mathbb{R}_+$ such that $\mathbb{P}(|M_{t\wedge\tau}| \leq C) = 1$ for any $t \geq 0$ then we have $\mathbb{E}(M_\tau) = \mathbb{E}(M_0)$.

Let's assume $X_0 = 0$. Then $M_0 = \exp(\alpha \cdot 0 - \lambda \cdot 0) = e^0 = 1$. So, $\mathbb{E}(M_\tau) = 1$. $\mathbb{E}[\exp(\alpha x - \lambda\tau)] = 1$. $\mathbb{E}[e^{\alpha x}e^{-\lambda\tau}] = 1$. $e^{\alpha x}\mathbb{E}[e^{-\lambda\tau}] = 1$. $\mathbb{E}[e^{-\lambda\tau}] = e^{-\alpha x}$.

Now we need to choose the correct root for α. Since X_t has an upward drift ($\mu \geq 0$), to hit a positive level x, X_t will tend to increase. Consider the case where $\alpha < 0$. As $t \to \infty$, if X_t does not hit x (which it does with probability 1), then $X_t \to \infty$. $M_t = \exp(\alpha X_t - \lambda t)$ would tend to 0 if $\alpha X_t \to -\infty$. Let $\alpha = -\mu - \sqrt{\mu^2 + 2\lambda}$. This value of α is negative. As $X_t \to \infty$, $e^{\alpha X_t} \to 0$. Also $e^{-\lambda t} \to 0$. The problem with using this α in the optional stopping theorem is the boundedness condition. For the martingale $M_t = \exp(\alpha B_t - \lambda t)$, for $\alpha < 0$, as $B_t \to \infty$, $M_t \to 0$. As $B_t \to -\infty$, $M_t \to \infty$. So M_t is not bounded.

However, we are hitting a level $x > 0$. When τ is the hitting time of x, $X_\tau = x$. The process $M_{t\wedge\tau}$ is $e^{\alpha X_{t\wedge\tau} - \lambda(t\wedge\tau)}$. For $t < \tau$, $0 \leq X_{t\wedge\tau} < x$ (if $x > 0$ and $X_0 = 0$). For $t \geq \tau$, $X_{t\wedge\tau} = x$. So, $0 \leq X_{t\wedge\tau} \leq x$. This means $\alpha X_{t\wedge\tau}$ is bounded by αx (if $\alpha > 0$) or 0 (if $\alpha < 0$). Let's consider $\alpha_1 = -\mu + \sqrt{\mu^2 + 2\lambda}$ and $\alpha_2 = -\mu - \sqrt{\mu^2 + 2\lambda}$.

If we use $\alpha_1 = -\mu + \sqrt{\mu^2 + 2\lambda}$: Since $\mu \geq 0$ and $\lambda > 0$, $\sqrt{\mu^2 + 2\lambda} > \sqrt{\mu^2} = |\mu|$. If $\mu = 0$, $\alpha_1 = \sqrt{2\lambda}$. If $\mu > 0$, α_1 is positive. So $\alpha_1 > 0$. In this case, $e^{\alpha_1 X_{t\wedge\tau}} \leq e^{\alpha_1 x}$ for $X_0 = 0, x > 0$. And $e^{-\lambda(t\wedge\tau)} \leq 1$. So $M_{t\wedge\tau}$ is bounded by $e^{\alpha_1 x}$. Thus the optional stopping theorem applies for α_1.

So, using $\alpha = -\mu + \sqrt{\mu^2 + 2\lambda}$: $\mathbb{E}[e^{-\lambda\tau}] = e^{-\alpha x} = e^{-x(-\mu + \sqrt{\mu^2 + 2\lambda})}$. $\mathbb{E}[e^{-\lambda\tau}] = \exp\left(x(\mu - \sqrt{\mu^2 + 2\lambda})\right)$.

This is the standard result for the Laplace transform of the hitting time of a Brownian motion with drift.