Midterm Exam (first midterm) - December 14, 2018, Stochastic Analysis

Family name ____________________________ **Given name** ____________________________

Signature ____________________________ **Neptun Code** ____________________________

No calculators or electronic devices are allowed. One formula sheet with 15 formulas is allowed.

1. Let $S_n = \xi_1 + \cdots + \xi_n$, where $\xi_1, \xi_2, \ldots,$ are i.i.d. and

$$\mathbb{P}(\xi_k = 1) = \mathbb{P}(\xi_k = -1) = \mathbb{P}(\xi_k = 0) = \frac{1}{3}, \qquad k \geq 1.$$

Let $(\mathcal{F}_n)_{n\geq 0}$ denote the natural filtration of (S_n). Let

$$X_n = S_n^2.$$

(a) (4 marks) Find the discrete Doob-Meyer decomposition of the process $(X_n)_{n\geq 1}$, i.e., write

$$X_n = A_n + M_n,$$

where (A_n) is a predictable process and (M_n) is a martingale with zero expectation. Give a simple and explicit formula for A_n.

(b) (3 marks) Let $\tau = \min\{\, n : |S_n| = 5\}$. Use the optional stopping theorem to calculate $\mathbb{E}(\tau)$. *Instruction:* You can use the optional stopping theorem without checking its conditions.

2. Denote by $(B(t))$ the standard Brownian motion. Let us define

$$Y_t = 4B(t) - tB(4), \qquad Z_t = (4-t)B\left(\frac{t}{1-t/4}\right), \qquad 0 \leq t < 4.$$

(a) (4 marks) Show that the stochastic processes $(Y_t)_{0\leq t<4}$ and $(Z_t)_{0\leq t<4}$ have the same law. *Hint:* You can use without proof that both $(Y_t)_{0\leq t<4}$ and $(Z_t)_{0\leq t<4}$ are Gaussian processes.

(b) (4 marks) Find the conditional expectation of Z_1 with respect to the sigma-algebra generated by the random variable Z_3.