Make-up midterm Exam - May 10, 2023, Stochastic Analysis

Family name ______________________ **Given name** ______________________

Signature ______________________ **Neptun Code** ______________________

No calculators or electronic devices are allowed. One formula sheet with 15 formulas is OK.

1. (8 points) Let $Y_1, Y_2, \dots$ denote i.i.d. random variables with distribution

$$\mathbb{P}(Y_k = -1) = \mathbb{P}(Y_k = 0) = \mathbb{P}(Y_k = 1) = 1/3. \tag{1}$$

Let $x_0 \in \mathbb{N}_+$. Let $X_n = x_0 + Y_1 + \cdots + Y_n$. Let $(\mathcal{F}_n)$ denote the natural filtration of (X_n).

(a) Let $M_n = X_n^3 - a \cdot n \cdot X_n$. How to choose $a \in \mathbb{R}$ if you want (M_n) to be a martingale?

(b) If $x \in \mathbb{Z}$, let $T_x := \min\{\, n \,:\, X_n = x \,\}$. Let $x_0 < R \in \mathbb{N}$. Find $\mathbb{E}(\, T_R \,|\, T_R < T_0 \,)$.
Help: Use the optional stopping theorem (OST) with an appropriately chosen stopping time. You can use the OST without checking that its conditions hold. You can also use without proof that $\mathbb{P}(T_R < T_0) = \frac{x_0}{R}$. You can also use that for any random variable X and any event A we have

$$\mathbb{E}(\, X \,|\, A \,) = \frac{\mathbb{E}(X \mathbb{1}[A])}{\mathbb{P}(A)}.$$

2. (7 points) Let (B_t) denote standard Brownian motion and let $(\mathcal{F}_t)$ denote its natural filtration. Let $X = \int_0^4 B_t \, \mathrm{d}B_t$. Calculate the conditional expectation of X^2 given $\mathcal{F}_2$.
Instruction: The final result should be simplified to the point that it does not contain conditional expectations or integrals.