Midterm Exam - April 20, 2023, Stochastic Analysis, Solutions

1. (7 points) Let X_n denote the position of a one-dimensional simple symmetric random walker at time n. Let $(\mathcal{F}_n)_{n\geq 0}$ denote the natural filtration of (X_n).

   (a) Given $a_k \in \mathbb{R}_+, k \in \mathbb{N}$, how to choose $b_k \in \mathbb{R}_+$, $k \in \mathbb{N}$ if we want (M_n) to be a martingale, where

$$M_n = \prod_{k=1}^{n} \frac{(a_k)^{X_k - X_{k-1}}}{b_k}, \qquad n = 0, 1, 2, \ldots \tag{1}$$

   (b) Write $M_n - 1$ as the discrete stochastic integral $(H \cdot X)_n$ of a predictable process (H_n) with respect to the martingale (X_n). Write the formula for H_n in a way that shows that (H_n) is indeed predictable.

Solution:

(a) (M_n) is clearly an adapted process. $M_0 = 1$ and $M_n = M_{n-1}\frac{(a_n)^{X_n - X_{n-1}}}{b_n}$, thus

$$\mathbb{E}[M_n \,|\, \mathcal{F}_{n-1}] \overset{(\Diamond)}{=} M_{n-1}\mathbb{E}\left[\frac{(a_n)^{X_n - X_{n-1}}}{b_n} \,\Big|\, \mathcal{F}_{n-1}\right] \overset{(\blacklozenge)}{=} M_{n-1}\mathbb{E}\left[\frac{(a_n)^{X_n - X_{n-1}}}{b_n}\right] = M_{n-1}\frac{\frac{1}{2}\frac{1}{a_n} + \frac{1}{2}a_n}{b_n}$$

where in $(\Diamond)$ we used that $M_{n-1} \in \mathcal{F}_{n-1}$ and in $(\blacklozenge)$ we used that the random walk has independent increments. Thus $b_k = \frac{1}{2}\frac{1}{a_k} + \frac{1}{2}a_k$.

(b) We want to identify the random variable H_n for which $M_n - M_{n-1} = H_n \cdot (X_n - X_{n-1})$. Now

$$M_n - M_{n-1} = M_{n-1}\left(\frac{(a_n)^{X_n - X_{n-1}}}{b_n} - 1\right) = \frac{M_{n-1}}{b_n}\left((a_n)^{X_n - X_{n-1}} - \frac{1}{2}\frac{1}{a_n} - \frac{1}{2}a_n\right) \overset{(**)}{=}$$
$$M_{n-1}\frac{a_n - \frac{1}{a_n}}{2b_n}(X_n - X_{n-1}),$$

where $(**)$ holds if $X_n - X_{n-1} = 1$ and also if $X_n - X_{n-1} = -1$ (these are the two possible values of $X_n - X_{n-1}$). Thus $H_n = M_{n-1}\frac{a_n - \frac{1}{a_n}}{2b_n}$, which is indeed $\mathcal{F}_{n-1}$-measurable since $M_{n-1} \in \mathcal{F}_{n-1}$ and $\frac{a_n - \frac{1}{a_n}}{2b_n}$ is deterministic.

2. (8 points) Let $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = t$. Let $L_n = \sum_{k=1}^{n} e^{B(t_{k-1})} \cdot (B(t_k) - B(t_{k-1}))$ and $\mathcal{I} = \int_0^t e^{B_s}\, \mathrm{d}B_s$. Show that

$$\mathbb{E}((\mathcal{I} - L_n)^2) = \sum_{k=1}^{n} e^{2t_{k-1}} \cdot \left[\frac{1}{2}\left(e^{2(t_k - t_{k-1})} - 1\right) - 4\left(e^{\frac{1}{2}(t_k - t_{k-1})} - 1\right) + (t_k - t_{k-1})\right]. \tag{2}$$

Help: If $X \sim \mathcal{N}(0, \sigma^2)$ then the moment generating function of X is $M(\lambda) = e^{\frac{1}{2}\lambda^2\sigma^2}$.

Solution: Let $X_t := \sum_{k=1}^{n} e^{B(t_{k-1})}\mathbb{1}[t_{k-1} < t \leq t_k]$, thus (X_t) is a simple predictable process. We have

$$\mathbb{E}((\mathcal{I} - L_n)^2) = \mathbb{E}\left[\left(\int_0^t (e^{B_s} - X_s)\, \mathrm{d}B_s\right)^2\right] \overset{(*)}{=} \mathbb{E}\left[\int_0^t (e^{B_s} - X_s)^2\, \mathrm{d}s\right] = \sum_{k=1}^{n}\int_{t_{k-1}}^{t_k} \mathbb{E}[(e^{B_s} - e^{B_{t_{k-1}}})^2]\, \mathrm{d}s,$$

where $(*)$ holds by the Itō isometry. Now for any $t_{k-1} \leq s \leq t_k$ we have

$$\mathbb{E}[(e^{B_s} - e^{B_{t_{k-1}}})^2] = \mathbb{E}[e^{2B_{t_{k-1}}}(e^{B_s - B_{t_{k-1}}} - 1)^2] \overset{(\circ)}{=} \mathbb{E}[e^{2B_{t_{k-1}}}]\mathbb{E}[(e^{B_s - B_{t_{k-1}}} - 1)^2] =$$
$$\mathbb{E}[e^{2B_{t_{k-1}}}]\mathbb{E}[e^{2(B_s - B_{t_{k-1}})} - 2e^{B_s - B_{t_{k-1}}} + 1] \overset{(\bullet)}{=} e^{\frac{1}{2}2^2 t_{k-1}}\left(e^{\frac{1}{2}2^2(s - t_{k-1})} - 2e^{\frac{1}{2}(s - t_{k-1})} + 1\right).$$

where $(\circ)$ holds since Brownian motion has independent increments and in $(\bullet)$ we used the *Help* concerning moment generating functions as well as $B_{t_{k-1}} \sim \mathcal{N}(0, t_{k-1})$ and $(B_s - B_{t_{k-1}}) \sim \mathcal{N}(0, s - t_{k-1})$. Now

$$\int_{t_{k-1}}^{t_k} e^{\frac{1}{2}2^2 t_{k-1}}\left(e^{\frac{1}{2}2^2(s - t_{k-1})} - 2e^{\frac{1}{2}(s - t_{k-1})} + 1\right)\mathrm{d}s =$$
$$e^{2t_{k-1}} \cdot \left[\frac{1}{2}\left(e^{2(t_k - t_{k-1})} - 1\right) - 4\left(e^{\frac{1}{2}(t_k - t_{k-1})} - 1\right) + (t_k - t_{k-1})\right],$$

and putting together all of the above gives the desired (2).