Midterm Exam - October 11, 2018, Stochastic Analysis, GROUP A

1. Let $Y_1, Y_2, \dots$ denote i.i.d. random variables with distribution

$$\mathbb{P}(Y_k = 1) = \frac{3}{4}, \qquad \mathbb{P}(Y_k = -1) = \frac{1}{4}.$$

Let $X_n = 1 + Y_1 + \cdots + Y_n$. Let $\mathcal{F}_n = \sigma(Y_1, \dots, Y_n)$. For $x \in \mathbb{Z}$, Let $T_x := \inf\{ n \,:\, X_n = x \}$.

(a) (2 marks) Find a constant $C \neq 1$ such that $\left(C^{X_n}\right)$ is a martingale.

(b) (2 marks) Give a formula for $q := \mathbb{P}(T_0 < T_6)$ using the optional stopping theorem.

(c) (2 marks) Find a constant m such that $(X_n - mn)$ is a martingale.

(d) (2 marks) Give a formula for $\mathbb{E}(\min\{T_0, T_6\})$ using the optional stopping theorem.

Solution:

(a) $\mathbb{E}(C^{X_n} \,|\, \mathcal{F}_{n-1}) = C^{X_{n-1}} \mathbb{E}(C^{Y_n}) = C^{X_{n-1}} \left(\frac{3}{4} C + \frac{1}{4}\frac{1}{C}\right)$. We want $\frac{3}{4}C + \frac{1}{4}\frac{1}{C} = 1$.
We want $\frac{3}{4}C^2 - C + \frac{1}{4} = 0$. Thus $C_{1,2} = \frac{1 \pm \sqrt{1 - 4\frac{3}{4}\frac{1}{4}}}{2\frac{3}{4}}$. Thus $C_1 = 1$ and $C_2 = \frac{1}{3}$.
Thus (3^{-X_n}) is a martingale.

(b) Let $\tau = \min\{T_0, T_6\}$. $C = C^{X_0} = \mathbb{E}(C^{X_\tau}) = qC^0 + (1-q)C^6 = C^6 + q(C^0 - C^6)$.
Thus $q = \frac{C - C^6}{1 - C^6}$.

(c) $\mathbb{E}(X_n - mn \,|\, \mathcal{F}_{n-1}) = X_{n-1} - m(n-1) + \mathbb{E}(Y_n - m) = X_{n-1} - m(n-1) + \frac{1}{2} - m$, so $m = \frac{1}{2}$.

(d) $1 = \mathbb{E}(X_0 - m0) = \mathbb{E}(X_\tau - m\tau) = \mathbb{E}(X_\tau) - m\mathbb{E}(\tau)$. Therefore:
$\mathbb{E}(\min\{T_0, T_6\}) = \mathbb{E}(\tau) = \frac{1}{m}\left(\mathbb{E}(X_\tau) - 1\right) = \frac{1}{m}\left(q \cdot 0 + (1-q)\cdot 6 - 1\right) = \frac{1}{m}\left(5 - 6q\right)$.

2. Let $(B(t))$ denote a standard Brownian motion.
Recall the definition of the stationary O.-U. process $X_t = e^{-2t} B(e^{4t})$.

(a) (2 marks) Find the covariance matrix of (X_{-1}, X_1).

(b) (2 marks) Calculate the conditional expectation of X_1 given the σ-algebra generated by X_{-1}.

(c) (3 marks) Calculate the conditional expectation of X_{-1} given the σ-algebra generated by X_1.

Solution:

(a) $\mathrm{Cov}(X_1, X_1) = e^{-4}\mathrm{Cov}(B(e^4), B(e^4)) = e^{-4}e^4 = 1$.
$\mathrm{Cov}(X_{-1}, X_{-1}) = e^4 \mathrm{Cov}(B(e^{-4}), B(e^{-4})) = e^4 e^{-4} = 1$.
$\mathrm{Cov}(X_{-1}, X_1) = e^2 e^{-2}\mathrm{Cov}(B(e^{-4}), B(e^4)) = e^{-4}$.
Thus $\underline{\underline{C}} = \begin{pmatrix} 1 & e^{-4} \\ e^{-4} & 1 \end{pmatrix}$.

(b)

$$\mathbb{E}\left(X_1 \,|X_{-1}\right) = \mathbb{E}\left(e^{-2}B(e^4) \,|\, e^2 B(e^{-4})\right) = e^{-2}\mathbb{E}\left(B(e^4) \,|\, B(e^{-4})\right) =$$
$$e^{-2}\mathbb{E}\left(B(e^{-4}) + (B(e^4) - B(e^{-4})) \,|\, B(e^{-4})\right) =$$
$$e^{-2}\mathbb{E}\left(B(e^{-4}) \,|\, B(e^{-4})\right) + e^{-2}\mathbb{E}\left(B(e^4) - B(e^{-4}) \,|\, B(e^{-4})\right) = e^{-2}B(e^{-4}) + 0 = e^{-4}X_{-1}$$

(c) Noting $\mathbb{E}(X_{-1}) = \mathbb{E}(X_1) = 0$ and taking a look at $\underline{\underline{C}}$ we see that the multivariate normal random vector (X_{-1}, X_1) has the same distribution as the multivariate normal random vector (X_1, X_{-1}). Therefore $\mathbb{E}\left(X_{-1} \,|X_1\right) = e^{-4}X_1$.