**Midterm Exam - October 11, 2018, Stochastic Analysis, GROUP B**

**Family name** ______________________ **Given name** ______________________

**Signature** ______________________ **Neptun Code** ______________________

**No calculators or electronic devices are allowed. One formula sheet with 15 formulas is allowed.**

1. Let $(B(t))$ denote a standard Brownian motion.
   Recall the definition of the stationary O.-U. process $Y_t = e^{-t}B(e^{2t})$.
   (a) (2 marks) Find the covariance matrix of $(Y_{-3}, Y_3)$.
   (b) (2 marks) Calculate the conditional expectation of $Y_3$ given the $\sigma$-algebra generated by $Y_{-3}$.
   (c) (3 marks) Calculate the conditional expectation of $Y_{-3}$ given the $\sigma$-algebra generated by $Y_3$.

   **Solution:**

   (a) $\mathrm{Cov}(Y_3, Y_3) = e^{-6}\mathrm{Cov}(B(e^6), B(e^6)) = e^{-6}e^6 = 1$.
   $\mathrm{Cov}(Y_{-3}, Y_{-3}) = e^{6}\mathrm{Cov}(B(e^{-6}), B(e^{-6})) = e^{6}e^{-6} = 1$.
   $\mathrm{Cov}(Y_{-3}, Y_3) = e^{3}e^{-3}\mathrm{Cov}(B(e^{-6}), B(e^6)) = e^{-6}$.
   Thus $\underline{\underline{C}} = \begin{pmatrix} 1 & e^{-6} \\ e^{-6} & 1 \end{pmatrix}$.
   (b)
   $$\mathbb{E}\left(Y_3 \,|Y_{-3}\right) = \mathbb{E}\left(e^{-3}B(e^6)\,|\,e^3B(e^{-6})\right) = e^{-3}\mathbb{E}\left(B(e^6)\,|\,B(e^{-6})\right) =$$
   $$e^{-3}\mathbb{E}\left(B(e^{-6}) + (B(e^6) - B(e^{-6}))\,|\,B(e^{-6})\right) =$$
   $$e^{-3}\mathbb{E}\left(B(e^{-6})\,|\,B(e^{-6})\right) + e^{-3}\mathbb{E}\left(B(e^6) - B(e^{-6})\,|\,B(e^{-6})\right) = e^{-3}B(e^{-6}) + 0 = e^{-6}Y_{-3}$$
   (c) Noting $\mathbb{E}(Y_{-3}) = \mathbb{E}(Y_3) = 0$ and taking a look at $\underline{\underline{C}}$ we see that the multivariate normal random vector $(Y_{-3}, Y_3)$ has the same distribution as the multivariate normal random vector $(Y_3, Y_{-3})$. Therefore $\mathbb{E}\left(Y_{-3}\,|Y_3\right) = e^{-6}Y_3$.

2. Let $X_1, X_2, \dots$ denote i.i.d. random variables with distribution
   $$\mathbb{P}(X_k = 1) = \frac{4}{5}, \qquad \mathbb{P}(X_k = -1) = \frac{1}{5}.$$
   Let $Z_n = 1 + X_1 + \dots + X_n$. Let $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$. For $x \in \mathbb{Z}$, Let $T_x := \inf\{\, n \,:\, Z_n = x\,\}$.
   (a) (2 marks) Find a constant $\alpha \neq 1$ such that $\left(\alpha^{Z_n}\right)$ is a martingale.
   (b) (2 marks) Give a formula for $p := \mathbb{P}(T_0 < T_7)$ using the optional stopping theorem.
   *Instruction:* You don't have to verify that the optional stopping theorem can be applied here.
   *Instruction:* It is enough to give a correct formula which contains $\alpha$ as a parameter.
   (c) (2 marks) Find a constant $\mu$ such that $(Z_n - \mu n)$ is a martingale.
   (d) (2 marks) Give a formula for $\mathbb{E}(\min\{T_0, T_7\})$ using the optional stopping theorem.
   *Instruction:* You don't have to verify that the optional stopping theorem can be applied here.
   *Instruction:* It is enough to give a correct formula which contains $p$ and $\mu$ as parameters.

   **Solution:**

   (a) $\mathbb{E}(\alpha^{Z_n}\,|\,\mathcal{F}_{n-1}) = \alpha^{Z_{n-1}}\mathbb{E}(\alpha^{X_n}) = \alpha^{Z_{n-1}}\left(\frac{4}{5}\alpha + \frac{1}{5}\frac{1}{\alpha}\right)$. We want $\frac{4}{5}\alpha + \frac{1}{5}\frac{1}{\alpha} = 1$.
   We want $\frac{4}{5}\alpha^2 - \alpha + \frac{1}{5} = 0$. Thus $\alpha_{1,2} = \frac{1 \pm \sqrt{1 - 4\frac{4}{5}\frac{1}{5}}}{2\frac{4}{5}}$. Thus $\alpha_1 = 1$ and $\alpha_2 = \frac{1}{4}$.
   Thus $(4^{-Z_n})$ is a martingale.
   (b) Let $\tau = \min\{T_0, T_7\}$. $\alpha = \alpha^{Z_0} = \mathbb{E}(\alpha^{Z_\tau}) = p\alpha^0 + (1-p)\alpha^7 = \alpha^7 + p(\alpha^0 - \alpha^7)$.
   Thus $p = \frac{\alpha - \alpha^7}{1 - \alpha^7}$.
   (c) $\mathbb{E}(Z_n - \mu n\,|\,\mathcal{F}_{n-1}) = Z_{n-1} - \mu(n-1) + \mathbb{E}(X_n - \mu) = Z_{n-1} - \mu(n-1) + \frac{3}{5} - \mu$, so $\mu = \frac{3}{5}$.
   (d) $1 = \mathbb{E}(Z_0 - \mu 0) = \mathbb{E}(Z_\tau - \mu\tau) = \mathbb{E}(Z_\tau) - \mu\mathbb{E}(\tau)$. Therefore:
   $\mathbb{E}(\min\{T_0, T_7\}) = \mathbb{E}(\tau) = \frac{1}{\mu}\left(\mathbb{E}(Z_\tau) - 1\right) = \frac{1}{\mu}\left(p \cdot 0 + (1-p)\cdot 7 - 1\right) = \frac{1}{\mu}\left(6 - 7p\right)$.