Midterm Exam (first midterm) - December 6, 2018, Stochastic Analysis

Family name ____________________________ **Given name** ____________________________

Signature ____________________________ **Neptun Code** ____________________________

No calculators or electronic devices are allowed. One formula sheet with 15 formulas is allowed.

1. Let (B_t) denote standard Brownian motion.

   (a) (4 marks) Let Z be a random variable with standard normal distribution which is independent from (B_t). Let $Y = aB_3 + bZ$. How to choose the positive constants a and b if we want (Y, B_3) to have the same joint distribution as (B_2, B_3)?

   (b) (2 marks) What is the conditional density function of B_2 if we condition on $B_3 = y$ for some $y \in \mathbb{R}$?

   (c) (1 mark) What is the conditional density function of B_3 if we condition on $B_2 = z$ for some $z \in \mathbb{R}$?

2. Let $S_n = \xi_1 + \cdots + \xi_n$, where $\xi_1, \xi_2, \ldots,$ are i.i.d. and $\mathbb{P}(\xi_k = 1) = \mathbb{P}(\xi_k = -1) = \frac{1}{2}$, $k \geq 1$. Let $(\mathcal{F}_n)_{n\geq 0}$ denote the natural filtration of (S_n).

   (a) (4 marks) Let $X_n = (2S_n - 1)^2$ for $n \in \mathbb{N}$. Find the discrete Doob-Meyer decomposition of the process $(X_n)_{n\geq 1}$, i.e., write
   $$(2S_n - 1)^2 = A_n + M_n,$$
   where (A_n) is a predictable process and (M_n) is a martingale with zero expectation. Give a simple and explicit formula for A_n.

   (b) (2 marks) Write M_n as the discrete stochastic integral $(H \cdot S)_n$ of a predictable process (H_n) with respect to the martingale (S_n). Explicitly state the formula for H_n.

   (c) (2 marks) Calculate $\mathrm{Var}(X_n)$ using the results of the previous sub-exercises and the Pythagorean theorem for square-integrable martingales.