Midterm Exam (first midterm) - December 12, 2018, Stochastic Analysis

Family name ____________________________ **Given name** ____________________________

Signature ____________________________ **Neptun Code** ____________________________

No calculators or electronic devices are allowed. One formula sheet with 15 formulas is allowed.

1. Let (X, Y) denote a pair of continuous random variables with joint density function

$$f(x,y) = c \exp\left(-\frac{5}{2}x^2 - 3xy - y^2\right).$$

   (a) (2 marks) Rewrite the joint density function in the form

$$f(\underline{x}) = \frac{1}{\sqrt{\det(\underline{\underline{C}})}} \frac{1}{\sqrt{2\pi}^n} \exp\left(-\frac{1}{2}(\underline{x}-\underline{m})^T \underline{\underline{C}}^{-1}(\underline{x}-\underline{m})\right)$$

   for some appropriate choice of n, $\underline{\underline{C}}$ and $\underline{m}$.

   (b) (1 marks) Find the normalizing constant c in the formula defining $f(x,y)$ above.

   (c) (2 marks) Find the density function $f_X(x)$ of X.

   (d) (3 marks) Find $\mathbb{E}(\,Y \,|\, \sigma(X)\,)$.

   Hint: All of the above tasks can be solved without calculating integrals: use the facts learnt about multivariate normal distribution in class.

2. Let $S_n = \xi_1 + \cdots + \xi_n$, where $\xi_1, \xi_2, \ldots,$ are i.i.d. and $\mathbb{P}(\xi_k = 1) = \mathbb{P}(\xi_k = -1) = \frac{1}{2}$, $k \geq 1$. Let $(\mathcal{F}_n)_{n\geq 0}$ denote the natural filtration of (S_n).

   (a) (3 marks) How to choose $C \in \mathbb{R}_+$ if we want $M_n = 2^{S_n}/C^n$ to be a martingale?

   (b) (4 marks) Write $M_n - M_0$ as the discrete stochastic integral $(H \cdot S)_n$ of a predictable process (H_n) with respect to the martingale (S_n). Explicitly state the formula for H_n.