Midterm Exam - November 22, 2018, Stochastic Analysis, GROUP A

Family name ________________ **Given name** ________________

Signature ________________ **Neptun Code** ________________

1. (a) (5 marks) Find the covariance of $\int_0^3 (2-7B_u)\,\mathrm{d}B_u$ and $\int_0^5 (2B_u^2-1)\,\mathrm{d}B_u$.
   (b) (2 marks) Find a simple closed formula for the value of $\int_0^3 (2-7B_u)\,\mathrm{d}B_u$.

   Solution:

   (a)

$$\mathrm{Cov}\left(\int_0^5 (2-7B_u)\mathbb{1}[u\le 3]\,\mathrm{d}B_u,\ \int_0^5 (2B_u^2-1)\,\mathrm{d}B_u\right) \overset{(*)}{=} \mathbb{E}\left(\int_0^3 (2-7B_u)(2B_u^2-1)\,\mathrm{d}u\right) =$$

$$\mathbb{E}\left(\int_0^3 4B_u^2-2-14B_u^3+7B_u\,\mathrm{d}u\right) \overset{(**)}{=} \int_0^3 4\mathbb{E}(B_u^2)-2-14\mathbb{E}(B_u^3)+7\mathbb{E}(B_u)\,\mathrm{d}u =$$

$$\int_0^3 4u-2-14\cdot 0+7\cdot 0\,\mathrm{d}u = \int_0^3 (4u-2)\,\mathrm{d}u = 12,$$

   where we used a formula from page 69 of the lecture notes in (*) and Fubini's theorem in (**).

   (b) $\int_0^3 (2-7B_u)\,\mathrm{d}B_u = 2\int_0^3 \mathrm{d}B_u - 7\int_0^3 B_u\,\mathrm{d}B_u = 2B_3 - 7\frac{1}{2}\left(B_3^2-3\right) = 2B_3 - \frac{7}{2}B_3^2 + \frac{21}{2}$.

2. Let us define

$$X_t = \frac{1}{\sqrt{4-t}}\exp\left(\frac{B_t^2}{2t-8}\right), \qquad 0\le t\le 2.$$

   (a) (4 marks) Show that $(X_t)_{0\le t\le 2}$ is a martingale.
   (b) (4 marks) Let $\mathcal{F}_t = \sigma(B_s,\, 0\le s\le t)$ denote the sigma-algebra generated by the Brownian motion up to time t. Find the constant C and the process $(Y_t)_{0\le t\le 2}$ adapted to the filtration $(\mathcal{F}_t)_{0\le t\le 2}$ such that

$$\exp\left(-\frac{1}{4}B_2^2\right) = C + \int_0^2 Y_s\,\mathrm{d}B_s.$$

   Solution:

   (a) We use the time-dependent Itō formula. Let $f(t,x) = (4-t)^{-1/2}\exp\left(\frac{-x^2/2}{4-t}\right)$, then $X_t = f(t,B_t)$.

   $f_x(t,x) = (4-t)^{-1/2}\exp\left(\frac{-x^2/2}{4-t}\right)\frac{-x}{4-t} = -(4-t)^{-3/2}x\exp(\dots)$

   $f_{xx}(t,x) = -(4-t)^{-3/2}\exp(\dots) + (4-t)^{-5/2}x^2\exp(\dots)$

$$f_t(t,x) = -\frac{1}{2}(4-t)^{-3/2}(-1)\exp(\dots) + (4-t)^{-1/2}\exp(\dots)\frac{-x^2/2}{(4-t)^2} =$$

$$\frac{1}{2}(4-t)^{-3/2}\exp(\dots) - \frac{1}{2}(4-t)^{-5/2}x^2\exp(\dots)$$

   Therefore we have $f_t(t,x)+\frac{1}{2}f_{xx}(t,x)\equiv 0$. By the time-dependent Itō formula, we have

$$\mathrm{d}X_t = \mathrm{d}f(t,B_t) = f_x(t,B_t)\,\mathrm{d}B_t + \left(f_t(t,B_t)+\frac{1}{2}f_{xx}(t,B_t)\right)\mathrm{d}t = f_x(t,B_t)\,\mathrm{d}B_t,$$

   the coefficient of $\mathrm{d}t$ vanishes, therefore (X_t) is a martingale.

   (b) $X_2 = X_0 + \int_0^2 \mathrm{d}X_t = X_0 + \int_0^2 f_x(t,B_t)\,\mathrm{d}B_t$. Now $X_2 = \frac{1}{\sqrt{2}}\exp\left(-\frac{1}{4}B_2^2\right)$ and $X_0 = \frac{1}{2}$, thus

$$\exp\left(-\frac{1}{4}B_2^2\right) = \sqrt{2}X_2 = \frac{1}{\sqrt{2}} + \int_0^2 \sqrt{2}f_x(t,B_t)\,\mathrm{d}B_t, \quad \text{thus} \quad Y_t = -\frac{\sqrt{2}B_t}{(4-t)^{3/2}}\exp\left(\frac{B_t^2}{2t-8}\right)$$