Midterm Exam (second midterm) - December 12, 2018, Stochastic Analysis

Family name ______________________ **Given name** ______________________

Signature ______________________ **Neptun Code** ______________________

No calculators or electronic devices are allowed. One formula sheet with 15 formulas is allowed.

1. Denote by $(B(t))$ the standard Brownian motion.

   (a) (5 marks) Find $\beta \in \mathbb{R}_+$ and $c > 0$ so that $(X(t))_{t\geq 0}$ and $(Y(t))_{t\geq 0}$ have the same law, where

   $$X(t) = B(c \cdot t^{\beta}), \qquad Y(t) = \int_0^t s^{-1/3}\,\mathrm{d}B(s).$$

   (b) (3 marks) Let $Z = \int_0^1 u^{2/3}\,\mathrm{d}Y(u)$, where $Y(t)$ is defined above. What is the distribution of Z?

2. We know that the Itō process (X_t) satisfies

   $$\mathrm{d}X_t = 2X_t\,\mathrm{d}t + \sqrt{X_t}\,\mathrm{d}B_t.$$

   Let $Y_t = \gamma X_t^{\alpha}$ for some $\alpha \in \mathbb{R}$ and $\gamma > 0$. We do not know the value of α and γ, but we do know that the the quadratic variation of (Y_t) satisfies $[Y]_t = t$ for all $t \geq 0$.

   (a) (3 marks) Find the value of α and γ.

   (b) (2 marks) Identify the function $g : \mathbb{R}_+ \to \mathbb{R}$ such that (M_t) is a martingale with $M_0 = 0$, where

   $$M_t = Y_t - Y_0 - \int_0^t g(Y_s)\,\mathrm{d}s.$$

   (c) (2 marks) The resulting process (M_t) is a famous stochastic process. Which one is it and why?

   Hint: First calculate the stochastic differential $\mathrm{d}Y_t$.