Second make-up midterm - December 7, 2016, 17.15-18.00, Stochastic Analysis

1. Let us define $g(x) = 2x + 1$. We approximate the stochastic integral $\mathcal{I} = \int_0^1 g(B(t))\mathrm{d}B(t)$ with the sum

$$\mathcal{I}_n = \sum_{k=1}^{n} g\left(B\left(\frac{k-1}{n}\right)\right) \cdot \left(B\left(\frac{k}{n}\right) - B\left(\frac{k-1}{n}\right)\right).$$

   (a) (1 marks) Find the simple predictable function $(X_n(t))$, $0 \le t \le 1$ for which $\mathcal{I}_n = \int_0^1 X_n(t)\mathrm{d}B(t)$.

   (b) (4 marks) What is the smallest value of n for which $\mathbb{E}\left((\mathcal{I} - \mathcal{I}_n)^2\right) \le \frac{1}{100}$ holds?

2. (5 marks) Find the Doob-Meyer decomposition $X_t = A_t + M_t$ of

$$X_t = e^{-4B_t - 8t},$$

   where A_t is an adapted process with finite total variation and M_t is a martingale. Write M_t as an Itō integral w.r.t. Brownian motion.

3. (5 marks) Let (X_t) and (Y_t) denote Itô processes that satisfy

$$X_t = 1 - \int_0^t Y_s \mathrm{d}B_s, \quad Y_t = \int_0^t X_s \mathrm{d}B_s.$$

   Find the exact value of $Z_t = X_t^2 + Y_t^2$ for any $t \ge 0$. *Hint:* First calculate $\mathrm{d}Z_t$.

Solutions.

1. (a) Let us define the partition $\Delta_n = \{t_0, \dots, t_n\}$ of the interval $[0,1]$ by setting $t_k = \frac{k}{n}$. Then

$$X_n(t) = \sum_{k=1}^{n} g(B(t_{k-1}))\mathbb{1}[t_{k-1} < t \le t_k].$$

   (b) We use the Itô isometry:

$$\mathbb{E}\left((\mathcal{I} - \mathcal{I}_n)^2\right) = \int_0^1 \mathbb{E}\left[(g(B_t) - X_n(t))^2\right] \mathrm{d}t = \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \mathbb{E}\left[(g(B(t)) - g(B(t_{k-1})))^2\right] \mathrm{d}t =$$

$$\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \mathbb{E}\left[(2(B(t) - B(t_{k-1})))^2\right] \mathrm{d}t = 4\sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} (t - t_{k-1})\,\mathrm{d}t = 4\sum_{k=1}^{n} \frac{1}{2}(t_k - t_{k-1})^2$$

   Now by $t_k = \frac{k}{n}$ we obtain $t_k - t_{k-1} = \frac{1}{n}$, thus $\mathbb{E}\left((\mathcal{I} - \mathcal{I}_n)^2\right) = 4\sum_{k=1}^{n} \frac{1}{2}\frac{1}{n^2} = \frac{2}{n}$.
   Thus the smallest value of n for which we have $\mathbb{E}\left((\mathcal{I} - \mathcal{I}_n)^2\right) \le \frac{1}{100}$ is $n = 200$.

2. $f(t,x) = e^{-4x-8t}$, $f_x = -4f$, $f_{xx} = 16f$, $f_t = -8f$. We use the time-dependent Itô formula:

$$\mathrm{d}f(t,B_t) = f_x(t,B_t)\mathrm{d}B_t + f_t(t,B_t)\mathrm{d}t + \frac{1}{2}f_{xx}(t,B_t)\mathrm{d}[B]_t =$$

$$f_x(t,B_t)\mathrm{d}B_t + (f_t(t,B_t) + \frac{1}{2}f_{xx}(t,B_t))\mathrm{d}t = f_x(t,B_t)\mathrm{d}B_t$$

   Thus $M_t = f(t,B_t)$ is a martingale, so $A_t \equiv 0$. $M_t = \int_0^t -4e^{-4x-8t}\mathrm{d}B_t$.

3. We have $\mathrm{d}X_t = -Y_t\mathrm{d}B_t$ and $\mathrm{d}Y_t = X_t\mathrm{d}B_t$ and thus

$$\mathrm{d}Z_t = \mathrm{d}X_t^2 + \mathrm{d}Y_t^2 = \left(2X_t\mathrm{d}X_t + \frac{1}{2}2\mathrm{d}[X]_t\right) + \left(2Y_t\mathrm{d}Y_t + \frac{1}{2}2\mathrm{d}[Y]_t\right) =$$

$$\left(2X_t(-Y_t)\mathrm{d}B_t + (-Y_t)^2\mathrm{d}t\right) + \left(2Y_tX_t\mathrm{d}B_t + (X_t)^2\mathrm{d}t\right) = (Y_t^2 + X_t^2)\mathrm{d}t = Z_t\mathrm{d}t$$

   Therefore $\frac{\mathrm{d}}{\mathrm{d}t}Z_t = Z_t$, i.e., $Z_t' = Z_t$ and also $Z_0 = 1^2 + 0^2 = 1$, so $Z_t = e^t$.