Midterm Exam - November 24, 2016, Stochastic Analysis

Family name ______________________ **Given name** ______________________

Signature ______________________ **Neptun Code** ______________________

No calculators or electronic devices are allowed. One formula sheet with 15 formulas is allowed.

1. (5 marks) Calculate the variance of $\int_0^3 e^{2B_t-2t}\mathrm{d}B_t$.

   Hint: The moment generating function of $X \sim \mathcal{N}(\mu,\sigma^2)$ is $M(\lambda) = e^{\mu\lambda+\frac{1}{2}\sigma^2\lambda^2}$.

2. (5 marks) Give an explicit formula for the cumulative distribution function $F(x) = \mathbb{P}(X \le x),\ x \in \mathbb{R}$ of the random variable X, where

$$X = \int_0^2 \frac{\mathrm{d}B_s}{1+B_s^2} - \int_0^2 \frac{B_s\mathrm{d}s}{(B_s^2+1)^2}.$$

   Hint: You may use $\Phi(\cdot)$, the c.d.f. of the standard normal distribution in your solution.

3. (5 marks) Let (B_t) and $(\widetilde{B}_t)$ denote i.i.d. standard Brownian motions. Show that $M_t = 4B_t^3\widetilde{B}_t - 4B_t\widetilde{B}_t^3$ is a martingale by writing it as $M_t = \int_0^t X_s\mathrm{d}B_s + \int_0^t \widetilde{X}_s\mathrm{d}\widetilde{B}_s$ for some adapted processes (X_t) and $(\widetilde{X}_t)$.

Solutions.

1. Let $Y = \int_0^3 e^{2B_t-2t}\mathrm{d}B_t$. We know $\mathbb{E}[Y] = 0$ so $\mathrm{Var}(Y) = \mathbb{E}[Y^2]$. We use Itô isometry:

$$\mathbb{E}[Y^2] = \mathbb{E}[\int_0^3 (e^{2B_t-2t})^2\mathrm{d}t] = \int_0^3 e^{-4t}\mathbb{E}[e^{4B_t}]\mathrm{d}t \overset{(*)}{=} \int_0^3 e^{4t}\mathrm{d}t = \frac{e^{12}-1}{4},$$

   where in $(*)$ we used that $B_t \sim \mathcal{N}(0,t)$, so $\mathbb{E}[e^{4B_t}] = M(4) = e^{\frac{1}{2}t4^2} = e^{8t}$.

   Alternative solution: we have learnt (see page 118) that $M_t = 1 + \int_0^t \lambda M_s\mathrm{d}B_s$ where $M_t = e^{\lambda B_t - \frac{1}{2}\lambda^2 t}$. Now $\lambda = 2$ and $t = 3$, so $Y = \int_0^3 M_t\mathrm{d}B_t = (M_3-1)/2$. Now $M_3 = e^{2B_3-6}$ and

$$\mathrm{Var}(Y) = \mathrm{Var}(\frac{M_3-1}{2}) = \frac{1}{4}\mathrm{Var}(M_3) = \frac{1}{4}(\mathbb{E}(M_3^2) - \mathbb{E}(M_3)^2) = \frac{e^{-12}e^{\frac{1}{2}3\cdot 4^2} - 1^2}{4} = \frac{e^{12}-1}{4}.$$

2. If $g(x) = \frac{1}{1+x^2}$ then $g'(x) = \frac{-2x}{(1+x^2)^2}$, so $X = \int_0^2 g(B_s)\mathrm{d}B_s + \int_0^2 \frac{1}{2}g'(B_s)\mathrm{d}s$.

   Now if $\int g(x)\mathrm{d}x = f(x) + c$ then $X = \int_0^2 f'(B_s)\mathrm{d}B_s + \frac{1}{2}\int_0^2 f''(B_s)\mathrm{d}s = f(B_2) - f(B_0)$ by Itô's formula. $f(x) = \arctan(x)$ and $\arctan(0) = 0$, thus $X = \arctan(B_2)$, thus for any $x \in (-\pi/2, \pi/2)$ we have

$$F(x) = \mathbb{P}(X \le x) = \mathbb{P}(\arctan(B_2) \le x) = \mathbb{P}(B_2 \le \tan(x)) = \mathbb{P}\left(\frac{B_2}{\sqrt{2}} \le \frac{\tan(x)}{\sqrt{2}}\right) = \Phi\left(\frac{\tan(x)}{\sqrt{2}}\right)$$

3. $M_t = f(B_t, \widetilde{B}_t)$ where $f(x,y) = 4x^3y - 4xy^3$.

$$f_x = 12x^2y - 4y^3, \qquad f_y = 4x^3 - 12xy^2, \qquad f_{xx} = 24xy, \qquad f_{yy} = -24xy, \qquad \Delta f = f_{xx} + f_{yy} \equiv 0.$$

   We use the multivariate Itô's formula:

$$\mathrm{d}M_t = \mathrm{d}f(B_t,\widetilde{B}_t) = f_x(B_t,\widetilde{B}_t)\mathrm{d}B_t + f_y(B_t,\widetilde{B}_t)\mathrm{d}\widetilde{B}_t + \frac{1}{2}\Delta f(B_t,\widetilde{B}_t)\mathrm{d}t,$$

   the coefficient of $\mathrm{d}t$ vanishes, thus M_t is a martingale and

$$M_t = \int_0^t (12B_s^2\widetilde{B}_s - 4\widetilde{B}_s^3)\mathrm{d}B_s + \int_0^t (4B_s^3 - 12B_s\widetilde{B}_s^2)\mathrm{d}\widetilde{B}_s$$

   Remark: $f(x,y)$ is harmonic because $f(x,y) = \mathrm{Im}(z^4) = \mathrm{Im}((x+iy)^4)$.