Second make-up midterm - December 20, 2016, Stochastic Analysis

Family name ______________________ **Given name** ______________________

Signature ______________________ **Neptun Code** ______________________

No calculators or electronic devices are allowed. One formula sheet with 15 formulas is allowed.

1. (5 marks) We say that two stochastic processes $(X(t))_{t\geq 0}$ and $(Y(t))_{t\geq 0}$ have the same law if for every choice of $n \geq 1$ and $0 \leq t_1 < t_2 < \cdots < t_n$ the joint distributions of $(X(t_1), X(t_2), \ldots, X(t_n))$ and $(Y(t_1), Y(t_2), \ldots, Y(t_n))$ are the same. Denote by $(B(t))$ the standard Brownian motion.

   Given $\alpha \in (-\frac{1}{2}, +\infty)$, find $\beta \in \mathbb{R}_+$ and $c > 0$ so that $(X(t))_{t\geq 0}$ and $(Y(t))_{t\geq 0}$ have the same law, where

   $$X(t) = B(c \cdot t^{\beta}), \qquad Y(t) = \int_0^t s^{\alpha}\, \mathrm{d}B(s).$$

   Briefly explain why $(X(t))_{t\geq 0}$ and $(Y(t))_{t\geq 0}$ have the same law using results seen in class.

2. Let us define $Y_t = \int_0^t \ln\left(\frac{1+t}{1+s}\right) \mathrm{d}B_s$ for any $t \geq 0$.

   (a) (3 marks) Show that (Y_t) is an *Itô process* by rewriting it in the form $Y_t = Y_0 + \int_0^t \mu_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}B_s$.

   (b) (2 marks) Calculate the quadratic variation $[Y]_t$.

3. (5 marks) How to choose the differentiable function $g : \mathbb{R} \to \mathbb{R}$ so that

   $$M_t = g(t)\cos(2B_t)$$

   is a martingale with $M_0 = 3$?