**Midterm Exam - May 7, 2025, Stochastic Analysis**

**Family name** ______________________ **Given name** ______________________

**Signature** ______________________ **Neptun Code** ______________________

**No calculators or electronic devices are allowed. One formula sheet with 15 formulas is OK.**

1. Let $(B_t)$ denote standard Brownian motion and $(\mathcal{F}_t)$ its natural filtration. Let us fix $x_0 \in \mathbb{R}$. Let
$$X = \mathbb{1}\left[B_3 \le x_0\right] = \begin{cases} 1 & \text{if} \quad B_3 \le x_0, \\ 0 & \text{if} \quad B_3 > x_0. \end{cases}$$
   (a) (3 points) Calculate the conditional expectation $M_t := \mathbb{E}(X \mid \mathcal{F}_t)$ for all $0 \le t \le 3$. *Hint:* The answer will be of form $f(t, B_t)$ (if we hide the dependence of the answer on the parameter $x_0$ from our notation).
   (b) (2 points) Check that $(M_t)$ is a martingale by calculating its stochastic differential.
   (c) (2 points) Find the adapted process $(\sigma_t)_{0\le t\le 3}$ for which $X = \mathbb{E}[X] + \int_0^3 \sigma_s \, \mathrm{d}B_s$.

2. Let us consider the Itô process that satisfies
$$X_t = X_0 + 2B_t + \int_0^t (2 + 4s - 2X_s) \, \mathrm{d}s, \qquad t \ge 0,$$
where $X_0 \sim \mathcal{N}(0,1)$ and $X_0$ is independent from the Brownian motion.
   (a) (1 point) Write down the differential form of the above integral equation (i.e., find the SDE that $(X_t)$ satisfies).
   (b) (4 points) Find the strong solution of the SDE.
   (c) (3 points) Identify the distribution of $X_t$ for each $t \ge 0$.