STOCH. ANAL. MIDTERM 2., 2025.05.07.

1) a) $M_t = \mathbb{P}(B_3 \le x_0 \mid \mathcal{F}_t)$ IF $B_t = x$ THEN

$$M_t = \mathbb{P}(B_3 \le x_0 \mid B_t = x) = \mathbb{P}(B_3 - B_t \le x_0 - x) =$$

$$= \mathbb{P}(\mathcal{N}(0, 3-t) \le x_0 - x) = \Phi\left(\frac{x_0 - x}{\sqrt{3-t}}\right) =: f(t,x)$$

THUS $M_t = \Phi\left(\frac{x_0 - B_t}{\sqrt{3-t}}\right) = f(t, B_t)$

b) $\Phi'(x) = \varphi(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}$ $\quad \varphi'(x) = -x \cdot \varphi(x)$

$$f_x(t,x) = \varphi\left(\frac{x_0 - x}{\sqrt{3-t}}\right) \cdot \frac{-1}{\sqrt{3-t}}$$

$$f_{xx}(t,x) = -\left(\frac{x_0 - x}{\sqrt{3-t}}\right) \cdot \varphi\left(\frac{x_0 - x}{\sqrt{3-t}}\right) \cdot \left(\frac{-1}{\sqrt{3-t}}\right)^2$$

$$= \frac{x - x_0}{(3-t)^{3/2}} \cdot \varphi\left(\frac{x_0 - x}{\sqrt{3-t}}\right)$$

$$f_t(t,x) = \varphi\left(\frac{x_0 - x}{\sqrt{3-t}}\right) \cdot \left(-\frac{1}{2}\right) \cdot \frac{x_0 - x}{(3-t)^{3/2}} \cdot (-1)$$

TIME-DEPENDENT ITÓ FORMULA

NOTE: $f_t(t,x) + \frac{1}{2} f_{xx}(t,x) \equiv 0$

$$dM_t = df(t, B_t) \stackrel{!}{=} \underbrace{\left(f_t(t,B_t) + \frac{1}{2} f_{xx}(t,B_t)\right)}_{0 \,\leftarrow\, \text{DRIFT VANISHES}} dt +$$

$$+ f_x(t, B_t)\, dB_t$$

c) $\underset{\substack{\parallel \\ \mathcal{X}}}{M_3} - \underset{\substack{\parallel \\ \mathbb{E}(\mathcal{X})}}{M_0} = \int_0^3 dM_s = \int_0^3 \underbrace{f_x(t, B_s)}_{\sigma_s} dB_s$

2 a) $dX_t = (2+4t-2X_t)dt + 2dB_t$

b) THIS IS A LINEAR S.D.E.

HOMOGENIZED : $dY_t = -2Y_t dt$.

THUS: $Y_t = C \cdot e^{-2t}$ THUS: $X_t = C_t \cdot e^{-2t}$

$$dX_t = e^{-2t} dC_t + C_t \cdot (-2) \cdot e^{-2t} dt$$

$$= e^{-2t} dC_t - 2X_t dt$$, THUS: WANT:

$$e^{-2t} dC_t = (2+4t)dt + 2dB_t$$

$$dC_t = e^{2t} \cdot (2+4t) dt + 2e^{2t} dB_t$$

$$C_t = C_0 + \underbrace{\int_0^t e^{2s} \cdot (2+4s) ds}_{= 2 \cdot e^{2t} \cdot t} + \int_0^t 2e^{2s} dB_s$$

$$X_t = C_t \cdot e^{-2t} = X_0 \cdot e^{-2t} + 2t + 2 \cdot e^{-2t} \cdot \int_0^t e^{2s} dB_s$$

c) X_t HAS NORMAL DISTRIBUTION

$\mathbb{E}(X_t) = 2t$ $Var(X_t) = e^{-4t} \cdot Var(X_0) +$

$4 \cdot e^{-4t} \cdot Var\left(\int_0^t e^{2s} dB_s\right) = e^{-4t} + 4 \cdot e^{-4t} \cdot \int_0^t e^{4s} ds =$

$= e^{-4t} + 4 \cdot e^{-4t} \cdot \left(\frac{e^{4t}-1}{4}\right) = 1$, THUS $\boxed{X_t \sim N(2t, 1)}$