Midterm Exam - May 23, 2025, Stochastic Analysis

Family name ____________________ **Given name** ____________________

Signature ____________________ **Neptun Code** ____________________

No calculators or electronic devices are allowed. One formula sheet with 15 formulas is OK.

1. Let (B_t) denote standard Brownian motion and $(\mathcal{F}_t)$ its natural filtration. Let

$$X = \exp\left(2\int_0^3 t\,\mathrm{d}B_t\right)$$

   (a) (3 points) Calculate the conditional expectation $M_t := \mathbb{E}(X \mid \mathcal{F}_t)$ for all $0 \le t \le 3$.
   Hint: The moment generating function of the $\mathcal{N}(0,\sigma^2)$ distribution is $e^{\frac{1}{2}\sigma^2\lambda^2}$.

   (b) (2 points) Check that (M_t) is a martingale by calculating its stochastic differential.

   (c) (2 points) Find the adapted process $(\sigma_t)_{0\le t\le 3}$ for which $X = \mathbb{E}[X] + \int_0^3 \sigma_s\,\mathrm{d}B_s$.

2. (8 points) Let us consider the Itô process that satisfies

$$X_t = 16 + 4\int_0^t X_s^{3/4}\,\mathrm{d}B_s + 6\int_0^t X_s^{1/2}\,\mathrm{d}s, \qquad t \ge 0.$$

   Let τ denote the first time when the value of this Itô process becomes 1. Let τ^* denote the first time when the value of this Itô process becomes 81. Calculate the probability of the event $\{\tau^* < \tau\}$ (i.e., that the process (X_t) hits 81 before it first hits 1).

   Hint: Find a function $g : \mathbb{R} \to \mathbb{R}$ such that $M_t := g(X_t)$ is a martingale.