Midterm Exam - May 23, 2025, Stochastic Analysis (Solved by Google Gemini)

Let's break down these stochastic analysis problems.

Solution to Exercise 1

We are given $X = \exp\left(2\int_0^3 t\,\mathrm{d}B_t\right)$.

Part (a) Calculate the conditional expectation $M_t := \mathbb{E}(X \mid \mathcal{F}_t)$ for all $0 \le t \le 3$.

Let $I_t = \int_0^t s\,\mathrm{d}B_s$. Then $X = \exp(2I_3)$. We want to calculate $M_t = \mathbb{E}[\exp(2I_3)|\mathcal{F}_t]$.

We can write $I_3 = I_t + \int_t^3 s\,\mathrm{d}B_s$. So, $X = \exp\left(2I_t + 2\int_t^3 s\,\mathrm{d}B_s\right) = \exp(2I_t)\exp\left(2\int_t^3 s\,\mathrm{d}B_s\right)$.

Since I_t is $\mathcal{F}_t$-measurable, we have: $M_t = \exp(2I_t)\mathbb{E}\left[\exp\left(2\int_t^3 s\,\mathrm{d}B_s\right)\middle|\mathcal{F}_t\right]$.

The stochastic integral $\int_t^3 s\,\mathrm{d}B_s$ is independent of $\mathcal{F}_t$ and is a Gaussian random variable. To find its variance, we use Itô isometry: $\mathbb{E}\left[\left(\int_t^3 s\,\mathrm{d}B_s\right)^2\right] = \int_t^3 s^2\,\mathrm{d}s = \left[\frac{s^3}{3}\right]_t^3 = \frac{3^3}{3} - \frac{t^3}{3} = 9 - \frac{t^3}{3}$. So, $\int_t^3 s\,\mathrm{d}B_s \sim \mathcal{N}\left(0, 9 - \frac{t^3}{3}\right)$.

Let $Y = \int_t^3 s\,\mathrm{d}B_s$. We need to calculate $\mathbb{E}[e^{2Y}]$. This is the moment generating function of Y evaluated at $\lambda = 2$. Using the hint, if $Y \sim \mathcal{N}(0, \sigma^2)$, then $\mathbb{E}[e^{\lambda Y}] = e^{\frac{1}{2}\sigma^2\lambda^2}$. Here, $\sigma^2 = 9 - \frac{t^3}{3}$ and $\lambda = 2$. So, $\mathbb{E}\left[\exp\left(2\int_t^3 s\,\mathrm{d}B_s\right)\right] = \exp\left(\frac{1}{2}\left(9 - \frac{t^3}{3}\right)(2^2)\right) = \exp\left(\frac{1}{2}\left(9 - \frac{t^3}{3}\right)4\right) = \exp\left(2\left(9 - \frac{t^3}{3}\right)\right) = \exp\left(18 - \frac{2t^3}{3}\right)$.

Therefore, for $0 \le t \le 3$: $M_t = \exp(2I_t)\exp\left(18 - \frac{2t^3}{3}\right) = \exp\left(2\int_0^t s\,\mathrm{d}B_s + 18 - \frac{2t^3}{3}\right)$.

For $t > 3$, $\mathcal{F}_3 \subseteq \mathcal{F}_t$. Since X is $\mathcal{F}_3$-measurable, $M_t = \mathbb{E}[X|\mathcal{F}_t] = X$ for $t \ge 3$. So, for $t \in [0,3]$, $M_t = \exp\left(2\int_0^t s\,\mathrm{d}B_s + 18 - \frac{2t^3}{3}\right)$. And for $t \ge 3$, $M_t = \exp\left(2\int_0^3 s\,\mathrm{d}B_s\right)$.

Part (b) Check that (M_t) is a martingale by calculating its stochastic differential.

Let $M_t = \exp\left(2\int_0^t s\,\mathrm{d}B_s + 18 - \frac{2t^3}{3}\right)$. Let $f(t, B_t) = \exp(Y_t)$ where $Y_t = 2\int_0^t s\,\mathrm{d}B_s + 18 - \frac{2t^3}{3}$.

We use Itô's Lemma. Let $G(t, y) = e^y$. Then $\frac{\partial G}{\partial t} = 0$, $\frac{\partial G}{\partial y} = e^y$, $\frac{\partial^2 G}{\partial y^2} = e^y$. First, let's find the differential of Y_t: $\mathrm{d}Y_t = 2t\,\mathrm{d}B_t - 2t^2\,\mathrm{d}t$. The quadratic variation of Y_t is $\mathrm{d}\langle Y\rangle_t = (2t)^2\,\mathrm{d}t = 4t^2\,\mathrm{d}t$.

Now apply Itô's Lemma to $M_t = e^{Y_t}$: $\mathrm{d}M_t = e^{Y_t}\,\mathrm{d}Y_t + \frac{1}{2}e^{Y_t}\,\mathrm{d}\langle Y\rangle_t$ $\mathrm{d}M_t = M_t(2t\,\mathrm{d}B_t - 2t^2\,\mathrm{d}t) + \frac{1}{2}M_t(4t^2\,\mathrm{d}t)$ $\mathrm{d}M_t = M_t(2t\,\mathrm{d}B_t - 2t^2\,\mathrm{d}t + 2t^2\,\mathrm{d}t)$ $\mathrm{d}M_t = M_t(2t\,\mathrm{d}B_t)$.

The drift term is zero. Therefore, M_t is a martingale. This holds for $t \in [0,3]$.

Part (c) Find the adapted process $(\sigma_t)_{0\le t\le 3}$ for which $X = \mathbb{E}[X] + \int_0^3 \sigma_s\,\mathrm{d}B_s$.

This is related to the martingale representation theorem. We know $M_t = \mathbb{E}[X|\mathcal{F}_t]$. We have $\mathrm{d}M_t = M_t(2t\,\mathrm{d}B_t)$, so $\mathrm{d}M_t = (2tM_t)\,\mathrm{d}B_t$. By integrating both sides from 0 to t: $M_t - M_0 = \int_0^t 2sM_s\,\mathrm{d}B_s$.

We need this for $t = 3$. So: $M_3 - M_0 = \int_0^3 2sM_s\,\mathrm{d}B_s$. We know $M_3 = X$ (since $t = 3$ is the upper limit of the integral in X). $M_0 = \mathbb{E}[X|\mathcal{F}_0] = \mathbb{E}[X]$ (since $\mathcal{F}_0$ is the trivial sigma-algebra).

So, $X - \mathbb{E}[X] = \int_0^3 2sM_s\,\mathrm{d}B_s$. Therefore, $X = \mathbb{E}[X] + \int_0^3 2sM_s\,\mathrm{d}B_s$. Comparing this with $X = \mathbb{E}[X] + \int_0^3 \sigma_s\,\mathrm{d}B_s$, we find $\sigma_s = 2sM_s$.

Substituting the expression for M_s: $\sigma_s = 2s\exp\left(2\int_0^s u\,\mathrm{d}B_u + 18 - \frac{2s^3}{3}\right)$. This process is adapted since M_s is adapted.

To calculate $\mathbb{E}[X]$: $\mathbb{E}[X] = M_0 = \exp\left(2\int_0^0 u\,\mathrm{d}B_u + 18 - \frac{2(0)^3}{3}\right) = \exp(0 + 18 - 0) = e^{18}$.

So, $X = e^{18} + \int_0^3 2s \exp\left(2\int_0^s u \,\mathrm{d}B_u + 18 - \frac{2s^3}{3}\right) \mathrm{d}B_s$. And $\sigma_s = 2s\exp\left(2\int_0^s u\,\mathrm{d}B_u + 18 - \frac{2s^3}{3}\right)$.

Solution to Exercise 2

We are given the Itô process: $\mathrm{d}X_t = 4X_s^{3/4}\,\mathrm{d}B_s + 6X_s^{1/2}\,\mathrm{d}s$, with $X_0 = 16$. We want to calculate $\mathbb{P}(\tau^* < \tau)$, where $\tau = \inf\{t \geq 0 : X_t = 1\}$ and $\tau^* = \inf\{t \geq 0 : X_t = 81\}$.

We need to find a function $g(x)$ such that $M_t = g(X_t)$ is a martingale. Let $g(X_t)$ be a martingale. By Itô's Lemma: $\mathrm{d}g(X_t) = g'(X_t)\,\mathrm{d}X_t + \frac{1}{2}g''(X_t)\,(\mathrm{d}X_t)^2$. Substitute $\mathrm{d}X_t$: $\mathrm{d}X_t = 4X_t^{3/4}\,\mathrm{d}B_t + 6X_t^{1/2}\,\mathrm{d}t$. $(\mathrm{d}X_t)^2 = (4X_t^{3/4})^2\,\mathrm{d}t = 16X_t^{3/2}\,\mathrm{d}t$.

$\mathrm{d}g(X_t) = g'(X_t)(4X_t^{3/4}\,\mathrm{d}B_t + 6X_t^{1/2}\,\mathrm{d}t) + \frac{1}{2}g''(X_t)(16X_t^{3/2}\,\mathrm{d}t)$ $\mathrm{d}g(X_t) = 4X_t^{3/4}g'(X_t)\,\mathrm{d}B_t + \left(6X_t^{1/2}g'(X_t) + 8X_t^{3/2}g''(X_t)\right)\,\mathrm{d}t$.

For $g(X_t)$ to be a martingale, the drift term must be zero: $6X_t^{1/2}g'(X_t) + 8X_t^{3/2}g''(X_t) = 0$. We can simplify by dividing by $X_t^{1/2}$ (assuming $X_t > 0$): $6g'(X_t) + 8X_tg''(X_t) = 0$. This is a second-order ordinary differential equation for $g(x)$: $8xg''(x) + 6g'(x) = 0$.

Let $h(x) = g'(x)$. Then $8xh'(x) + 6h(x) = 0$. $\frac{h'(x)}{h(x)} = -\frac{6}{8x} = -\frac{3}{4x}$. Integrate both sides with respect to x: $\ln|h(x)| = -\frac{3}{4}\ln|x| + C_1 = \ln(x^{-3/4}) + C_1$. $h(x) = Cx^{-3/4}$.

Now, integrate $h(x)$ to find $g(x)$: $g(x) = \int Cx^{-3/4}\,\mathrm{d}x = C\frac{x^{1/4}}{1/4} + D = 4Cx^{1/4} + D$. Let's choose $C = 1/4$ and $D = 0$ for simplicity (or any other constants, as martingales are closed under linear transformations). So, let $g(x) = x^{1/4}$.

Let's check if $g(X_t) = X_t^{1/4}$ is a martingale. $g'(x) = \frac{1}{4}x^{-3/4}$ $g''(x) = -\frac{3}{16}x^{-7/4}$

Substitute into the drift term equation: $6x^{1/2}\left(\frac{1}{4}x^{-3/4}\right) + 8x^{3/2}\left(-\frac{3}{16}x^{-7/4}\right) = \frac{6}{4}x^{1/2-3/4} - \frac{24}{16}x^{3/2-7/4} = \frac{3}{2}x^{-1/4} - \frac{3}{2}x^{-1/4} = 0$. So, $g(X_t) = X_t^{1/4}$ is indeed a martingale.

Now we can use the optional stopping theorem. Let $M_t = X_t^{1/4}$. We are interested in the probability that X_t hits 81 before 1. The initial value is $X_0 = 16$. The values we are interested in are $a = 1$ and $b = 81$. Let $M_0 = X_0^{1/4} = 16^{1/4} = 2$. When X_t hits 1, M_t hits $1^{1/4} = 1$. When X_t hits 81, M_t hits $81^{1/4} = 3$.

Let $\tau_{\text{stop}} = \min(\tau, \tau^*)$. This is a stopping time. By the optional stopping theorem, $\mathbb{E}[M_{\tau_{\text{stop}}}] = M_0$. $M_{\tau_{\text{stop}}}$ can take two values: 1 (if $\tau < \tau^*$) or 3 (if $\tau^* < \tau$). Let $p = \mathbb{P}(\tau^* < \tau)$. Then $\mathbb{E}[M_{\tau_{\text{stop}}}] = 1 \cdot \mathbb{P}(\tau < \tau^*) + 3 \cdot \mathbb{P}(\tau^* < \tau)$. $M_0 = 2$. $1 \cdot (1-p) + 3 \cdot p = 2$. $1 - p + 3p = 2$. $1 + 2p = 2$. $2p = 1$. $p = \frac{1}{2}$.

Therefore, the probability that the process (X_t) hits 81 before it first hits 1 is $\frac{1}{2}$.