Second Midterm Exam - May 28, 2025, Stochastic Analysis

Family name ________________________ **Given name** ________________________

Signature ________________________ **Neptun Code** ________________________

No calculators or electronic devices are allowed. One formula sheet with 15 formulas is OK.

1. (8 points) Let $\underline{B}_t = (B_1(t), B_2(t), B_3(t), B_4(t))$ denote 4-dimensional Brownian motion started from $\underline{B}_0 = \underline{x}_0 \in \mathbb{R}^4$, $\underline{x}_0 \neq \underline{0}$. Let

$$X_t = (B_1(t))^2 + (B_2(t))^2 + (B_3(t))^2 + (B_4(t))^2.$$

   (a) Find $\alpha \in \mathbb{R} \setminus \{0\}$ such that the drift term of $M_t := (X_t)^\alpha$ vanishes.

   (b) Let $0 < r_1 < r_2 \in \mathbb{R}_+$ such that $\underline{x}_0$ is inside the ball $\mathcal{B}(r_2)$ of radius r_2 around the origin, but $\underline{x}_0$ is outside of the ball $\mathcal{B}(r_1)$ of radius r_1 around the origin. What is the probability that $(\underline{B}_t)$ exits $\mathcal{B}(r_2)$ before it enters $\mathcal{B}(r_1)$?

2. (7 points) Let $y_0 \in \mathbb{R}_+$. I know that $Y_t = y_0 + \int_0^t 4Y_u \,\mathrm{d}B_u - \int_0^t Y_u \,\mathrm{d}u$ holds for all $t \geq 0$ and I also know that $\mathbb{E}(Y_2) = e$.

   (a) What is y_0?

   (b) Show that $Z_t := (Y_t)^3$ is a time-homogeneous Itô diffusion process by finding the functions $\mu : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}$ for which $Z_t = Z_0 + \int_0^t \sigma(Z_s) \,\mathrm{d}B_s + \int_0^t \mu(Z_s) \,\mathrm{d}s$ holds for all $t \geq 0$.