Second Midterm Exam - May 28, 2025, Stochastic Analysis

Let's solve these stochastic analysis exercises.

Exercise 1: 4-dimensional Brownian motion

Let $\underline{B}_t = (B_1(t), B_2(t), B_3(t), B_4(t))$ denote 4-dimensional Brownian motion started from $\underline{B}_0 = \underline{x}_0 \in \mathbb{R}^4$, $\underline{x}_0 \neq \underline{0}$. Let $X_t = (B_1(t))^2 + (B_2(t))^2 + (B_3(t))^2 + (B_4(t))^2 = \|\underline{B}_t\|^2$.

Part (a): Find $\alpha \in \mathbb{R} \setminus \{0\}$ such that the drift term of $M_t := (X_t)^\alpha$ vanishes.

First, let's find the SDE for X_t. Since $\underline{B}_t$ is a 4-dimensional Brownian motion, each $B_i(t)$ is a standard 1-dimensional Brownian motion, and they are independent. For each $B_i(t)$, we have $dB_i(t) = 1 \cdot dB_i(t) + 0 \cdot dt$. Let $f(x_1, x_2, x_3, x_4) = x_1^2 + x_2^2 + x_3^2 + x_4^2$. Using Itô's Lemma for multi-dimensional processes: $dX_t = \sum_{i=1}^4 \frac{\partial f}{\partial x_i} dB_i(t) + \frac{1}{2} \sum_{i=1}^4 \frac{\partial^2 f}{\partial x_i^2} dt$ $\frac{\partial f}{\partial x_i} = 2x_i$ $\frac{\partial^2 f}{\partial x_i^2} = 2$ So, $dX_t = \sum_{i=1}^4 2B_i(t)dB_i(t) + \frac{1}{2}\sum_{i=1}^4 2dt = 2\sum_{i=1}^4 B_i(t)dB_i(t) + 4dt$.

Now, let $M_t = (X_t)^\alpha$. Let $g(x) = x^\alpha$. Using Itô's Lemma: $dM_t = g'(X_t)dX_t + \frac{1}{2}g''(X_t)(dX_t)^2$ $g'(x) = \alpha x^{\alpha-1}$ $g''(x) = \alpha(\alpha-1)x^{\alpha-2}$ We know $dX_t = 2\sum_{i=1}^4 B_i(t)dB_i(t) + 4dt$. $(dX_t)^2 = \left(2\sum_{i=1}^4 B_i(t)dB_i(t)\right)^2 = 4\sum_{i=1}^4 (B_i(t))^2(dB_i(t))^2 = 4\sum_{i=1}^4 (B_i(t))^2 dt = 4X_t dt$.

Substituting these into the Itô's Lemma for M_t: $dM_t = \alpha X_t^{\alpha-1}\left(2\sum_{i=1}^4 B_i(t)dB_i(t) + 4dt\right) + \frac{1}{2}\alpha(\alpha-1)X_t^{\alpha-2}(4X_t dt)$ $dM_t = 2\alpha X_t^{\alpha-1}\sum_{i=1}^4 B_i(t)dB_i(t) + 4\alpha X_t^{\alpha-1}dt + 2\alpha(\alpha-1)X_t^{\alpha-1}dt$ $dM_t = 2\alpha X_t^{\alpha-1}\sum_{i=1}^4 B_i(t)dB_i(t) + [4\alpha + 2\alpha(\alpha-1)]X_t^{\alpha-1}dt$ The drift term is $[4\alpha + 2\alpha(\alpha-1)]X_t^{\alpha-1}$. We want the drift term to vanish, so we set it to zero: $4\alpha + 2\alpha(\alpha-1) = 0$ Since $\alpha \in \mathbb{R} \setminus \{0\}$, we can divide by α: $4 + 2(\alpha-1) = 0$ $4 + 2\alpha - 2 = 0$ $2\alpha + 2 = 0$ $2\alpha = -2$ $\alpha = -1$.

Thus, for $\alpha = -1$, the drift term of $M_t = (X_t)^{-1}$ vanishes.

Part (b): Probability of exiting $\mathcal{B}(r_2)$ before entering $\mathcal{B}(r_1)$.

We are given $0 < r_1 < r_2 \in \mathbb{R}_+$ such that $\underline{x}_0$ is inside the ball $\mathcal{B}(r_2)$ (i.e., $\|\underline{x}_0\| < r_2$) and outside of the ball $\mathcal{B}(r_1)$ (i.e., $\|\underline{x}_0\| > r_1$). We want to find the probability that $(\underline{B}_t)$ exits $\mathcal{B}(r_2)$ before it enters $\mathcal{B}(r_1)$. This means we are looking for the probability that $\|\underline{B}_t\|$ reaches r_2 before it reaches r_1.

Let $X_t = \|\underline{B}_t\|^2$. From part (a), we know that $M_t = X_t^{-1}$ has a zero drift term. This means M_t is a local martingale. Let $M_t = X_t^{-1} = (\|\underline{B}_t\|^2)^{-1} = \|\underline{B}_t\|^{-2}$. Let $R_t = \|\underline{B}_t\|$. Then $M_t = R_t^{-2}$.

The stopping times are: $T_1 = \inf\{t \geq 0 : \|\underline{B}_t\| = r_1\}$ $T_2 = \inf\{t \geq 0 : \|\underline{B}_t\| = r_2\}$ We want to find $P(T_2 < T_1)$.

Since the process M_t is a local martingale and the process stopped at T is bounded, by the Optional Stopping Theorem, for almost surely finite stopping times T, $E[M_T] = M_0$. Let $T = \min(T_1, T_2)$. Since R_t starts between r_1 and r_2, X_t starts between r_1^2 and r_2^2. The process X_t will eventually hit either r_1^2 or r_2^2. So T is a finite stopping time. Also, M_t is bounded on the interval $[r_1^2, r_2^2]$, so M_T is bounded. Therefore, $E[M_T] = M_0$.

Let $p = P(T_2 < T_1)$. Then $P(T_1 < T_2) = 1 - p$. If $T_2 < T_1$, then $X_T = r_2^2$. If $T_1 < T_2$, then $X_T = r_1^2$.

$E[M_T] = E[M_T | T_2 < T_1]P(T_2 < T_1) + E[M_T | T_1 < T_2]P(T_1 < T_2)$ $M_0 = (\|\underline{x}_0\|)^{-2}$. $E[M_T] = (r_2^2)^{-1}p + (r_1^2)^{-1}(1-p)$ So, $\|\underline{x}_0\|^{-2} = r_2^{-2}p + r_1^{-2}(1-p)$. Let $x_0 = \|\underline{x}_0\|$. $x_0^{-2} = r_2^{-2}p + r_1^{-2} - r_1^{-2}p$ $x_0^{-2} - r_1^{-2} = p(r_2^{-2} - r_1^{-2})$ $p = \frac{x_0^{-2} - r_1^{-2}}{r_2^{-2} - r_1^{-2}}$

This is the probability that $(\underline{B}_t)$ exits $\mathcal{B}(r_2)$ before it enters $\mathcal{B}(r_1)$.

Exercise 2: Itô Diffusion Process

Let $y_0 \in \mathbb{R}_+$. $Y_t = y_0 + \int_0^t 4Y_u \, \mathrm{d}B_u - \int_0^t Y_u \, \mathrm{d}u$. We also know that $\mathbb{E}(Y_2) = e$.

Part (a): What is y_0?

The SDE for Y_t is $dY_t = 4Y_t dB_t - Y_t dt$. This is a geometric Brownian motion type SDE. Consider $dY_t = \sigma Y_t dB_t + \mu Y_t dt$. In our case, $\sigma = 4$ and $\mu = -1$. The solution to such an SDE is given by $Y_t = Y_0 \exp\left((\mu - \frac{1}{2}\sigma^2)t + \sigma B_t\right)$. So, $Y_t = y_0 \exp\left((-1 - \frac{1}{2}(4)^2)t + 4B_t\right)$ $Y_t = y_0 \exp\left((-1 - \frac{1}{2}(16))t + 4B_t\right)$ $Y_t = y_0 \exp\left((-1-8)t + 4B_t\right)$ $Y_t = y_0 \exp\left(-9t + 4B_t\right)$.

Now we need to find $\mathbb{E}(Y_2)$. $\mathbb{E}(Y_2) = \mathbb{E}\left[y_0 \exp\left(-9(2) + 4B_2\right)\right]$ $\mathbb{E}(Y_2) = y_0 \exp(-18)\mathbb{E}\left[\exp(4B_2)\right]$.

We know that for a standard normal random variable $Z \sim N(0,1)$, $E[e^{\lambda Z}] = e^{\frac{1}{2}\lambda^2}$. Here B_2 is a normal random variable with mean 0 and variance 2. So $B_2 \sim N(0,2)$. We can write $B_2 = \sqrt{2}Z$ where $Z \sim N(0,1)$. So, $\mathbb{E}\left[\exp(4B_2)\right] = \mathbb{E}\left[\exp(4\sqrt{2}Z)\right]$. Here $\lambda = 4\sqrt{2}$. $\mathbb{E}\left[\exp(4\sqrt{2}Z)\right] = \exp\left(\frac{1}{2}(4\sqrt{2})^2\right) = \exp\left(\frac{1}{2}(16 \cdot 2)\right) = \exp\left(\frac{1}{2}(32)\right) = \exp(16)$.

Therefore, $\mathbb{E}(Y_2) = y_0 \exp(-18)\exp(16) = y_0 \exp(-2)$. We are given that $\mathbb{E}(Y_2) = e$. So, $y_0 \exp(-2) = e$. $y_0 e^{-2} = e^1$. $y_0 = e^1 e^2 = e^3$. Thus, $y_0 = e^3$.

Part (b): Show that $Z_t := (Y_t)^3$ is a time-homogeneous Itô diffusion process by finding the functions $\mu : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}$.

Let $Z_t = (Y_t)^3$. Let $f(y) = y^3$. We use Itô's Lemma. $f'(y) = 3y^2$ $f''(y) = 6y$ From the SDE for Y_t, we have $dY_t = 4Y_t dB_t - Y_t dt$. So, $\sigma_Y(Y_t) = 4Y_t$ and $\mu_Y(Y_t) = -Y_t$. Itô's Lemma states: $dZ_t = f'(Y_t)dY_t + \frac{1}{2}f''(Y_t)(dY_t)^2$. Substitute dY_t: $dZ_t = 3Y_t^2(4Y_t dB_t - Y_t dt) + \frac{1}{2}(6Y_t)(4Y_t dB_t - Y_t dt)^2$. First, calculate $(dY_t)^2$: $(dY_t)^2 = (4Y_t dB_t - Y_t dt)^2 = (4Y_t dB_t)^2 = 16Y_t^2(dB_t)^2 = 16Y_t^2 dt$.

Now substitute back into the Itô's Lemma equation for dZ_t: $dZ_t = 12Y_t^3 dB_t - 3Y_t^3 dt + \frac{1}{2}(6Y_t)(16Y_t^2 dt)$ $dZ_t = 12Y_t^3 dB_t - 3Y_t^3 dt + 48Y_t^3 dt$ $dZ_t = 12Y_t^3 dB_t + (48Y_t^3 - 3Y_t^3)dt$ $dZ_t = 12Y_t^3 dB_t + 45Y_t^3 dt$.

We want to express Z_t in the form $dZ_t = \sigma(Z_t)dB_t + \mu(Z_t)dt$. Since $Z_t = Y_t^3$, we can substitute Y_t^3 with Z_t: $dZ_t = 12Z_t dB_t + 45Z_t dt$.

So, we have: $\sigma(Z_t) = 12Z_t$ $\mu(Z_t) = 45Z_t$

The functions $\mu(z) = 45z$ and $\sigma(z) = 12z$ depend only on Z_t and not explicitly on time t. Therefore, Z_t is a time-homogeneous Itô diffusion process. In particular, Z_t is also a geometric Brownian motion.