Midterm Exam (second midterm) - December 14, 2018, Stochastic Analysis

Family name ______________________________ **Given name** ______________________________

Signature ______________________________ **Neptun Code** ______________________________

No calculators or electronic devices are allowed. One formula sheet with 15 formulas is allowed.

1. Let (B_t) denote the standard Brownian motion. Let us define the process (X_t) by

$$\mathrm{d}X_t = 2 \cdot \mathrm{d}B_t - \mathrm{d}t, \qquad X_0 = 1.$$

Given some $\alpha, \beta \in \mathbb{R}$, let us define

$$Y_t := \exp\left(\beta X_t - \alpha t\right).$$

  (a) (3 marks) Calculate the stochastic differential $\mathrm{d}Y_t$.

  (b) (2 marks) Given $\alpha \in \mathbb{R}$ how should we choose $\beta \in \mathbb{R}$ if we want (Y_t) to be a martingale?
  Warning: Maybe there is no such value of β or maybe there are multiple such values of β, depending on the value of α.

  (c) (2 marks) Let $\tau = \min\{\, t \;:\; X_t = 0\,\}$ the hitting time of level 0. Apply the optional stopping theorem to calculate $\mathbb{E}\left(e^{-\tau}\right)$.
  Hint: You may assume without proof that $\mathbb{P}(\tau < +\infty) = 1$. You can use the following continuous-time form of the optional stopping theorem: if (M_t) is a martingale, τ is a stopping time satisfying $\mathbb{P}(\tau < +\infty) = 1$ and if there exists a constant $C \in \mathbb{R}_+$ such that $\mathbb{P}(|M_{t\wedge\tau}| \leq C) = 1$ for any $t \geq 0$ then we have $\mathbb{E}(M_\tau) = \mathbb{E}(M_0)$.

2. Let us define

$$Y_t = 2e^{-8t} + 4\int_0^t e^{8(s-t)}\,\mathrm{d}B_s$$

  (a) (4 marks) Calculate the stochastic differential of Y_t and show that (Y_t) is an Itô process by explicitly writing down the formula of the processes (μ_t) and (σ_t) for which

$$Y_t = Y_0 + \int_0^t \mu_u\,\mathrm{d}u + \int_0^t \sigma_u\,\mathrm{d}B_u.$$

  More specifically, find the functions $\widetilde{\mu} : \mathbb{R} \to \mathbb{R}$ and $\widetilde{\sigma} : \mathbb{R} \to \mathbb{R}$ such that $\mu_u = \widetilde{\mu}(Y_u)$ and $\sigma_u = \widetilde{\sigma}(Y_u)$ in the above formula.

  (b) (1 mark) Calculate the quadratic variation $[Y]_t$ on the interval $[0, t]$.

  (c) (3 marks) Find the distribution of Y_2.