Midterm Exam - May 25, 2023, Stochastic Analysis

1. Let (B_t) denote standard Brownian motion and $(\mathcal{F}_t)$ its natural filtration. Let $X := (B_4)^3$.

   (a) Calculate the conditional expectation $M_t := \mathbb{E}(X \mid \mathcal{F}_t)$ for all $0 \le t \le 4$.

   (b) Find the adapted process $(\sigma_t)_{0\le t\le 4}$ for which $X = \mathbb{E}[X] + \int_0^4 \sigma_s \, \mathrm{d}B_s$.

   Solution:

   (a) Let us fix $0 \le t \le 4$. Let $Y := B_4 - B_t$.

$$M_t = \mathbb{E}((B_4)^3 \mid \mathcal{F}_t) = \mathbb{E}((B_t + Y)^3 \mid \mathcal{F}_t) =$$
$$\mathbb{E}(B_t^3 \mid \mathcal{F}_t) + 3\mathbb{E}(B_t^2 Y \mid \mathcal{F}_t) + 3\mathbb{E}(B_t Y^2 \mid \mathcal{F}_t) + \mathbb{E}(Y^3 \mid \mathcal{F}_t) = B_t^3 + B_t^2 \cdot 0 + 3B_t(4-t) + 0 =$$
$$B_t^3 + 3B_t(4-t).$$

   (b) $(M_t)_{0\le t\le 4}$ is a martingale, $M_0 = \mathbb{E}[X] = 0$, $M_4 = X$. Let us write M_t as an Itō integral.

$$\mathrm{d}M_t = 3B_t^2 \, \mathrm{d}B_t + \frac{1}{2} 6B_t \, \mathrm{d}t + 3B_t(-1)\, \mathrm{d}t + 3(4-t)\, \mathrm{d}B_t = (3B_t^2 + 12 - 3t)\, \mathrm{d}B_t$$

   Integrating this we obtain $M_4 - M_0 = \int_0^4 (3B_t^2 + 12 - 3t)\, \mathrm{d}B_t$, thus $\sigma_t = 3B_t^2 + 12 - 3t$.

2. Let us consider the Itô process that satisfies

$$X_t = 5 + 3\int_0^t X_s \, \mathrm{d}s + 2\int_0^t X_s \, \mathrm{d}B_s, \qquad t \ge 0. \tag{1}$$

   (a) Find the value of $x \in \mathbb{R}$ for which $\mathbb{P}(X_4 \le x) = \frac{1}{2}$.

   (b) Let $Y_t = \sqrt{X_t}$. Show that (Y_t) is a time-homogeneous Itô diffusion process by writing down the drift coefficient $\mu : \mathbb{R} \to \mathbb{R}$ and the diffusion coefficient $\sigma : \mathbb{R} \to \mathbb{R}$ for which $\mathrm{d}Y_t = \mu(Y_t)\mathrm{d}t + \sigma(Y_t)\mathrm{d}B_t$.

   Solution: Taking the stochastic differential of both sides of (1), we obtain

$$\mathrm{d}X_t = 3X_t \, \mathrm{d}t + 2X_t \, \mathrm{d}B_t, \qquad X_0 = 5. \tag{2}$$

   This is the SDE of a geometric Brownian motion with parameters $r = 3$ and $\sigma = 2$,

   (a) thus we know from class that

$$X_t = 5\exp\left(2B_t + (3 - \frac{1}{2}2^2)t\right) = 5\exp(2B_t + t), \quad \text{in particular} \quad X_4 = 5\exp(2B_4 + 4).$$

   We have $\mathbb{P}(B_4 \le 0) = \frac{1}{2}$, thus if $x = 5 \cdot e^4$ then $\mathbb{P}(X_4 \le x) = \frac{1}{2}$.

   (b) Let us take the stochastic differential of Y_t:

$$\mathrm{d}Y_t = \mathrm{d}\,(X_t)^{1/2} = \frac{1}{2}(X_t)^{-1/2}\mathrm{d}X_t + \frac{1}{2}\cdot\frac{1}{2}\cdot(-\frac{1}{2})\cdot(X_t)^{-3/2}\,\mathrm{d}\,[X_t] \overset{(2)}{=}$$
$$\frac{1}{2}(X_t)^{-1/2}(3X_t\,\mathrm{d}t + 2X_t\,\mathrm{d}B_t) - \frac{1}{8}\cdot(X_t)^{-3/2}(4X_t^2)\,\mathrm{d}t = Y_t\,\mathrm{d}B_t + Y_t\,\mathrm{d}t.$$

   Thus $\sigma(x) = x$, $\mu(x) = x$, thus Y_t is also a geometric Brownian motion (with parameters $\sigma = r = 1$).