**Stoch. Anal. Exam, 18.12.2014**

*Info:* Each of the 4 questions is worth 12.5 marks. Write your name and Neptun code on each piece of paper that you submit. Separate the solutions of different exercises (and sub-exercises) with a horizontal line. No calculators or electronic devices are allowed. One hand-written A4-sized formula sheet is allowed. You have 110 minutes to complete this exam.

1. Let $(B_t)$ denote standard Brownian motion.
   (a) Find the cumulative distribution function of
   $$X = \min_{0 \le t \le 3} B_t,$$
   i.e., calculate $\mathbb{P}(X \le x)$ for all $x \in \mathbb{R}$. Please express your answer using the cumulative distribution function $\Phi(\cdot)$ of the standard normal random variable. Please explain how you used the *reflection principle* in your calculation by drawing a picture where the relevant quantities and random variables are clearly indicated.
   (b) Calculate the probability density function of $\tau = \min\{t \,:\, B_t = -3\}$.
2. (a) Given a stochastic process $(Y_t)$ define the notion of the *total variation* of $(Y_t)$ on $[0, T]$.
   (b) Given a stochastic process $(Y_t)$ define the notion of the *quadratic variation* of $(Y_t)$ on $[0, T]$.
   (c) State and prove the result about the quadratic variation of a process $(Y_t)_{0 \le t \le T}$ which has continuous trajectories and finite total variation on $[0, T]$.
   (d) State and prove the result about the characterization of the process $Y_t = \int_0^t X_s \, \mathrm{d}B_s$ which has finite total variation. Carefully state the theorems that you use in the proof.
   (e) Define the notion of an *Itō process* $(Y_t)$. This definition involves a decomposition of form
   $$Y_t = A_t + M_t.$$
   Prove that this decomposition is uniquely determined by the process $(Y_t)$.
3. Given two independent Brownian motions $(B_1(t))$ and $(B_2(t))$ construct a third Brownian motion $(B_3(t))$ such that the *mutual variation* $[B_1, B_3]_t$ is equal to $\sin(t)$ for all $t \ge 0$. Carefully state the theorems that you use to show that $(B_3(t))$ is indeed a Brownian motion and that $[B_1, B_3]_t$ is indeed equal to $\sin(t)$.
4. Let $(\underline{B}_t) = (B_1(t), \dots, B_4(t))$ denote the 4-dimensional Brownian motion started from $\underline{x}_0 \ne \underline{0}$. Let
   $$X_t = \|\underline{B}_t\|^{-2} = \frac{1}{B_1^2(t) + \cdots + B_4^2(t)}.$$
   (a) Calculate the stochastic differential $\mathrm{d}X_t$.
   (b) Prove that $X_{t \wedge \tau}$ is a martingale where $\tau = \min\{t \,:\, \|\underline{B}_t\| = \varepsilon\}$ and $0 < \varepsilon < \|\underline{x}_0\|$.
   (c) Prove that $X_t$ is not a martingale.