Stoch. Anal. Exam, December 20, 2016

Info: Each of the 4 questions is worth 12.5 marks. Write your name and Neptun code on each piece of paper that you submit. Separate the solutions of different exercises (and sub-exercises) with a horizontal line. No calculators or electronic devices are allowed. One hand-written two-sided A4-sized formula sheet with 30 formulas is allowed. You have 100 minutes to complete this exam.

1. Recall the stochastic process $B^{(N)}(t)$, $0 \le t \le 1$ from Paul Lévy's construction of Brownian motion:

$$B^{(N)}(t) = tX + \sum_{n=0}^{N-1} \sum_{k=1}^{2^n} a_n f_k^n(t) Y_k^n,$$

where X and Y_k^n, $0 \le n$, $1 \le k \le 2^n$ are independent standard normal random variables.

   (a) Draw a picture of $f_1^0(t)$, $f_1^1(t)$, $f_2^1(t)$ and the general $f_k^n(t)$, where $0 \le n$ and $1 \le k \le 2^n$.

   (b) Draw a picture of $B^{(2)}(t)$ and show on the same picture how we obtain $B^{(3)}(t)$ from $B^{(2)}(t)$.

   (c) Prove by induction on N that we can choose $a_1, a_2, \dots \in \mathbb{R}_+$ such that for each $N \ge 0$ the increments

$$B^{(N)}\left(\frac{1}{2^N}\right) - B^{(N)}\left(\frac{0}{2^N}\right), \quad B^{(N)}\left(\frac{2}{2^N}\right) - B^{(N)}\left(\frac{1}{2^N}\right), \quad \dots, \quad B^{(N)}\left(\frac{2^N}{2^N}\right) - B^{(N)}\left(\frac{2^N-1}{2^N}\right)$$

   are i.i.d. with distribution $\mathcal{N}\left(0, \frac{1}{2^N}\right)$. Derive the explicit formula for a_N, $N \ge 0$ and explain which properties of multivariate normal distribution you used in your proof.

2. We say that two stochastic processes $(X(t))_{t\ge 0}$ and $(Y(t))_{t \ge 0}$ have the same law if for every choice of $n \ge 1$ and $0 \le t_1 < t_2 < \dots < t_n$ the joint distributions of $(X(t_1), X(t_2), \dots, X(t_n))$ and $(Y(t_1), Y(t_2), \dots, Y(t_n))$ are the same. Denote by $(B(t))$ the standard Brownian motion.

   Given $\beta \in (-\frac{1}{2}, +\infty)$, find $\alpha \in \mathbb{R}_+$ and $c > 0$ so that $(X(t))_{t\ge 0}$ and $(Y(t))_{t\ge 0}$ have the same law, where

$$X(t) = B(c \cdot t^\alpha), \qquad Y(t) = \int_0^t s^\beta \, \mathrm{d}B(s).$$

   Briefly explain why $(X(t))_{t\ge 0}$ and $(Y(t))_{t\ge 0}$ have the same law using results seen in class.

3. For every $n \in \mathbb{N}$, let $\Delta_n = \{t_0, t_1, \dots, t_n\}$ denote a partition of the interval $[0,t]$, in other words, let $0 = t_0 < t_1 < \dots < t_n = t$. Denote by $|\Delta_n| = \max_{1\le k \le n}(t_k - t_{k-1})$. We assume that $\lim_{n\to\infty} |\Delta_n| = 0$.

   Let $g : \mathbb{R} \to \mathbb{R}$ denote a bounded and continuous function. Let (B_t) denote standard Brownian motion. Let us define

$$\mathcal{B}_n = \sum_{k=1}^{n} g(B_{t_{k-1}}) \cdot \left(B_{t_k} - B_{t_{k-1}}\right)^2.$$

   Prove that $\mathcal{B}_n$ converges in probability to $\int_0^t g(B_s)\mathrm{d}s$ as $n \to \infty$. Explain where you used the definition of Riemann integral, the Pythagorean theorem for square integrable martingales and Chebyshev's inequality in your proof.

4. Let $d \in \mathbb{R}_+$, $d \ne 2$. Let (Y_t) be a d-dimensional Bessel process, i.e., the solution of the SDE

$$\mathrm{d}Y_t = \frac{d-1}{2}\frac{\mathrm{d}t}{Y_t} + \mathrm{d}B_t, \qquad Y_0 = y_0 \in \mathbb{R}_+.$$

   (a) Find an $f : \mathbb{R} \to \mathbb{R}$ such that $f(Y_t)$ is a (local) martingale, i.e. the drift term of $\mathrm{d}f(Y_t)$ vanishes.

   (b) Use the optional stopping theorem to calculate $\mathbb{P}(T_a < T_b)$ where

$$0 < a \le y_0 \le b, \qquad T_a = \min\{t \,:\, Y_t = a\}, \qquad T_b = \min\{t \,:\, Y_t = b\}$$

   (c) Find $g : \mathbb{R} \to \mathbb{R}$ such that $g(Y_t) - t$ is a (local) martingale and calculate $\mathbb{E}(\tau)$, where $\tau = T_a \wedge T_b$.