Stoch. Anal. Exam, December 18, 2018

Info: Each of the 4 questions is worth 12.5 marks. Write your name and Neptun code on each piece of paper that you submit. Separate the solutions of different exercises (and sub-exercises) with a horizontal line. No calculators or electronic devices are allowed. One hand-written two-sided A4-sized formula sheet with 30 formulas is allowed. You have 90 minutes to complete this exam.

1. Let (B_t) denote standard Brownian motion. Given $x \geq 0$ and $t \geq 0$ let us define
$$M_t := \max_{0 \leq s \leq t} B_s, \qquad T_x = \min\{\, t \,:\, B_t = x \,\}.$$
   (a) (8 marks) Use the reflection principle to derive a formula for the cumulative distribution function of M_t for some fixed $t > 0$. Please draw a picture explaining how you used the reflection principle.

   (b) (4.5 marks) Calculate the probability density function of T_x for some fixed $x > 0$.

2. We approximate the stochastic integral $\mathcal{Z} = \int_0^1 (B(t) - 1)\mathrm{d}B(t)$ with the sum
$$\mathcal{Z}_n = \sum_{k=1}^{n} \left(B\left(\frac{k-1}{n}\right) - 1 \right) \cdot \left(B\left(\frac{k}{n}\right) - B\left(\frac{k-1}{n}\right) \right).$$
   (a) (3 marks) Write down the simple predictable function $(X_n(t))$, $0 \leq t \leq 1$ for which
$$\mathcal{Z}_n = \int_0^1 X_n(t)\mathrm{d}B(t).$$
   (b) (9.5 marks) What is the smallest value of n for which $\mathbb{E}\left[(\mathcal{Z} - \mathcal{Z}_n)^2\right] \leq 10^{-3}$ holds?

3. (12.5 marks) Paul Lévy's characterization of Brownian motion states that if (M_t) is a continuous martingale with $M_0 = 0$ and its quadratic variation satisfies $[M]_t = t$ for all $t \geq 0$, then (M_t) is actually a standard Brownian motion. Prove Paul Lévy's characterization of Brownian motion.

   Instruction: In your argument you can use without proof that if $\mathcal{F}$ is a sigma-algebra and X is a random variable and
$$\mathbb{E}\left(e^{iuX} \,\middle|\, \mathcal{F} \right) = e^{-\frac{1}{2}u^2\sigma^2} \qquad \text{for every} \quad u \in \mathbb{R}$$
   (where i denotes the complex imaginary unit) then X is independent from $\mathcal{F}$ and X has $\mathcal{N}(0, \sigma^2)$ distribution. When you prove Paul Lévy's characterization of Brownian motion, you may also use the simplifying assumption that the stochastic process (M_t) is an Itō process driven by multivariate Brownian motion.

   Hint: First show that for any $u \in \mathbb{R}$, the process (Z_t) defined by $Z_t = \exp\left(iuM_t + \frac{1}{2}u^2 t\right)$ is a martingale.

4. Let us consider the two dimensional Bessel process (X_t), i.e., the solution of the SDE
$$\mathrm{d}X_t = \frac{1}{2}\frac{1}{X_t}\,\mathrm{d}t + \mathrm{d}B_t, \qquad X_0 = x_0 > 0.$$
   (a) (8.5 marks) Find a function $f : \mathbb{R}_+ \to \mathbb{R}$ such that the process $(f(X_t))$ has zero drift.

   Hint: Calculate the stochastic differential $\mathrm{d}f(X_t)$ and then solve the resulting ODE for f.

   (b) (4 marks) Let $T_x = \min\{\, t \,:\, X_t = x \,\}$. Let $0 < a < x_0 < b < \infty$. Calculate $\mathbb{P}(T_a < T_b)$.

   Hint: Use the optional stopping theorem.