Stoch. Anal. Exam, June 15, 2023

Info: Each of the 4 questions is worth 12.5 marks. Write your name and Neptun code on each piece of paper that you submit. Separate the solutions of different exercises (and sub-exercises) with a horizontal line. No calculators or electronic devices are allowed. One hand-written two-sided A4-sized formula sheet with 30 formulas is allowed. You have 90 minutes to complete this exam.

1. (a) (4 points) Define the mutual variation of two stochastic processes.

   (b) (4 points) State and prove the result about the mutual variation of a continuous function and a function with finite total variation.

   (c) (4.5 points) State the formula about the mutual variation of two Itō processes driven by the same d-dimensional Brownian motion and give a heuristic proof using stochastic differentials.

2. (a) (6.5 points) Let $f : \mathbb{R}_+ \to \mathbb{R}$ and $g : \mathbb{R}_+ \to \mathbb{R}$ denote deterministic continuous functions. Let

$$M_t = \exp\left(i \int_0^t f(s)\,\mathrm{d}B_s + g(t) \right).$$

   Given f how to choose g if we want (M_t) to be a martingale satisfying $M_0 = 1$?

   (b) (6 points) Let $X = \exp\left(i \int_0^T f(s)\,\mathrm{d}B_s \right)$. Find $\mathbb{E}(X)$ and the adapted process $(\sigma(t))_{t=0}^T$ for which

$$X = \mathbb{E}(X) + \int_0^T \sigma(t)\,\mathrm{d}B_t.$$

3. Let (X_t) and (Y_t) denote Itō processes satisfying $\mathbb{E}\left(\int_0^T X_s^2\,\mathrm{d}s\right) < +\infty$ and $\mathbb{E}\left(\int_0^T Y_s^2\,\mathrm{d}s\right) < +\infty$ and

$$X_t = x_0 + \int_0^t \mu(X_s)\,\mathrm{d}s + \int_0^t \sigma(X_s)\,\mathrm{d}B_s, \qquad 0 \le t \le T, \tag{1}$$

$$Y_t = y_0 + \int_0^t \mu(Y_s)\,\mathrm{d}s + \int_0^t \sigma(Y_s)\,\mathrm{d}B_s, \qquad 0 \le t \le T. \tag{2}$$

   Let us assume that $|\mu(x) - \mu(y)| \le K \cdot |x - y|$ and $|\sigma(x) - \sigma(y)| \le K \cdot |x - y|$ for all $x, y \in \mathbb{R}$. Let us define $\Psi_t := \mathbb{E}[(X_t - Y_t)^2]$ for all $0 \le t \le T$.

   (a) (3 points) Show that $\mathbb{E}\left[\left(\int_0^t (\mu(X_s) - \mu(Y_s))\,\mathrm{d}s\right)^2\right] \le K^2 T \int_0^t \Psi_s\,\mathrm{d}s$, $0 \le t \le T$.

   (b) (3 points) Show that $\mathbb{E}\left[\left(\int_0^t (\sigma(X_s) - \sigma(Y_s))\,\mathrm{d}B_s\right)^2\right] \le K^2 \int_0^t \Psi_s\,\mathrm{d}s$, $0 \le t \le T$.

   (c) (3 points) State (but do not prove) Grönwall's inequality.

   (d) (3.5 points) Show that $\mathbb{E}[(X_t - Y_t)^2] \le 9(x_0 - y_0)^2 \exp\left(9K^2(T+1)t\right)$ holds for all $0 \le t \le T$.

4. *Poisson's equation with Dirichlet boundary condition.* Given a compact domain $\mathcal{D} \subseteq \mathbb{R}^d$ with a smooth boundary $\partial\mathcal{D}$, moreover $g : \mathcal{D} \to \mathbb{R}$ and $f : \partial\mathcal{D} \to \mathbb{R}$ (both bounded and continuous), we say that the function $u : \mathcal{D} \to \mathbb{R}$ is the solution of Poisson's equation with Dirichlet boundary condition if $\frac{1}{2}\Delta u(\underline{x}) = g(\underline{x})$ for all $\underline{x} \in \mathcal{D}^o$ (the interior of $\mathcal{D}$) and $u(\underline{x}) = f(\underline{x})$ for all $\underline{x} \in \partial\mathcal{D}$. Apply the optional stopping theorem (OST) to an appropriately chosen martingale to show that the solution u has the following probabilistic representation:

$$u(\underline{x}) = \mathbb{E}\left(f(\underline{B}_\tau) - \int_0^\tau g(\underline{B}_s)\,\mathrm{d}s \,\middle|\, \underline{B}_0 = \underline{x} \right), \qquad \underline{x} \in \mathcal{D},$$

   where $(\underline{B}_t)$ is a d-dim. Brownian motion and $\tau = \min\{\, t : \underline{B}_t \in \partial\mathcal{D}\}$ is the exit time of $(\underline{B}_t)$ from $\mathcal{D}^o$.

   Instruction: You do not have to check the conditions of the OST, just use it.