Stoch. Anal. Exam, 06.01.2015

Info: Each of the 4 questions is worth 12.5 marks. Write your name and Neptun code on each piece of paper that you submit. Separate the solutions of different exercises (and sub-exercises) with a horizontal line. No calculators or electronic devices are allowed. One hand-written A4-sized formula sheet is allowed. You have 110 minutes to complete this exam.

1. The aim of this exercise is to prove that if the stochastic process $(X_t)_{0\le t\le T}$ is left-continuous, adapted and $\mathbb{E}\left(\int_0^T X_t^2\,\mathrm{d}t\right) < +\infty$, and if $Y_t = \int_0^t X_s \mathrm{d}B_s$, $0 \le t \le T$, then

   (a) $(Y_t)_{0\le t\le T}$ is continuous and

   (b) $(Y_t)_{0\le t\le T}$ is a martingale.

   In your solution you can use the following facts without proving them, however you are asked to clearly indicate where you used them:

   (i) If $(\widetilde{X}_t)_{0\le t\le T}$ is a *simple predictable process* and $\mathbb{E}\left(\int_0^T \widetilde{X}_t^2\,\mathrm{d}t\right) < +\infty$, and if $\widetilde{Y}_t = \int_0^t \widetilde{X}_s \mathrm{d}B_s$, $0 \le t \le T$, then $(\widetilde{Y}_t)_{0\le t\le T}$ is a continuous martingale.

   (ii) If $(X_t)_{0\le t\le T}$ is left-continuous, adapted and $\mathbb{E}\left(\int_0^T X_t^2\,\mathrm{d}t\right) < +\infty$ then for every $\varepsilon > 0$ there exists a simple predictable process $(\widetilde{X}_t)_{0\le t\le T}$ such that $\mathbb{E}\left(\int_0^T (X_t - \widetilde{X}_t)^2\,\mathrm{d}t\right) < \varepsilon$.

   (iii) Itō isometry; submartingale inequality and its corollaries; the uniform limit of continuous functions is a continuous function; Borel-Cantelli lemma; Jensen's inequality for conditional expectations.

2. Let $S_n = \xi_1 + \cdots + \xi_n$, where $\xi_1, \xi_2, \ldots,$ are i.i.d. and $\mathbb{P}(\xi_k = 1) = \mathbb{P}(\xi_k = -1) = \frac{1}{2}$, $k \ge 1$. Let $(\mathcal{F}_n)_{n\ge 0}$ denote the natural filtration of (S_n), i.e., $\mathcal{F}_n = \sigma(S_1, \ldots, S_n) = \sigma(\xi_1, \ldots, \xi_n)$.

   (a) Show that any martingale (M_n) with $M_0 = 0$ adapted to $(\mathcal{F}_n)_{n\ge 0}$ is a discrete stochastic integral $(H \cdot S)_n$ of a predictable process (H_n) with respect to the martingale (S_n). Explicitly state the formula for H_n.

   (b) Show that if X is an $\mathcal{F}_{100}$-measurable random variable then

   $$X = \mathbb{E}[X] + \sum_{k=1}^{100} Y_k \cdot (S_k - S_{k-1})$$

   for some random variables $Y_1, \ldots, Y_{100}$, where Y_k is $\mathcal{F}_{k-1}$-measurable, $k = 1, \ldots, 100$.

3. If $\underline{x} = (x, y) \in \mathbb{R}^2$, denote by $\|\underline{x}\| = \sqrt{x^2 + y^2}$ the Euclidean norm of $\underline{x}$.

   The aim of this exercise is to show that if $\underline{B}_t = (B_1(t), B_2(t))$ is a 2-dimensional Brownian motion starting from $\underline{x}_0 \ne \underline{0}$ and if $T_x = \inf\{t \,:\, \|\underline{B}_t\| = x\}$ then $\mathbb{P}(T_0 = +\infty) = 1$, i.e., $\underline{B}_t$ never hits the origin $\underline{0}$.

   (a) Calculate the stochastic differential $\mathrm{d}M_t$ of

   $$M_t = \ln\left(\|\underline{B}_t\|\right).$$

   (b) Use the optional stopping theorem to calculate $\mathbb{P}[T_a < T_b]$ for any $0 < a < \|\underline{x}_0\| < b < +\infty$.

   (c) Show that $\mathbb{P}(T_0 = +\infty) = 1$. How did you use the continuity of $(\underline{B}_t)$?

4. Find the explicit solution of the stochastic differential equation

   $$\mathrm{d}X_t = \alpha X_t \mathrm{d}t + \beta X_t \mathrm{d}B_t, \qquad X_0 = \gamma, \qquad \alpha, \beta, \gamma \in \mathbb{R}_+$$