Stoch. Anal. Exam, January 3, 2017

Info: Each of the 4 questions is worth 12.5 marks. Write your name and Neptun code on each piece of paper that you submit. Separate the solutions of different exercises (and sub-exercises) with a horizontal line. No calculators or electronic devices are allowed. One hand-written two-sided A4-sized formula sheet with 30 formulas is allowed. You have 100 minutes to complete this exam.

1. (a) Define the notion of a *simple predictable process* (X_t).
   (b) Define $\int_0^T X_t \mathrm{d}B_t$ for a simple predictable process (X_t).
   (c) Prove that $\mathbb{E}\left[\int_0^T X_t \mathrm{d}B_t\right] = 0$ if (X_t) is a simple predictable process.
   (d) State and prove the *Itô isometry* for the simple predictable process (X_t).
   Instruction: You are allowed to use the *Pythagorean theorem for square integrable martingales* without proving it.

2. Let us define

$$p_n(t,x) := \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} \exp(\lambda x - \frac{\lambda^2}{2}t)\bigg|_{\lambda=0}, \qquad n = 0, 1, 2, \ldots$$

   (a) We say that the function $u(t,x)$ is a solution of the reverse heat equation if

$$\frac{\partial}{\partial t} u(t,x) \equiv -\frac{1}{2}\frac{\partial^2}{\partial x^2} u(t,x).$$

   Show that for any value of the parameter $\lambda \in \mathbb{R}$, the function $\exp(\lambda x - \frac{\lambda^2}{2}t)$ solves the reverse heat equation and that the polynomials $p_n(t,x), n \in \mathbb{N}$ inherit this property.
   (b) Show that for any $n \geq 0$ the process defined by $M_n(t) := p_n(t, B_t)$ is a martingale.

3. Let $(B_1(t), \ldots, B_n(t))$ denote n-dimensional Brownian motion. Prove that $X_t = \sqrt{B_1^2(t) + \cdots + B_n^2(t)}$ is a *weak solution* of the SDE

$$\mathrm{d}X_t = \frac{n-1}{2}\frac{1}{X_t}\mathrm{d}t + \mathrm{d}B_t.$$

   Instruction: You should also explain why this solution is *weak* and how you used Paul Lévy's characterisation of Brownian motion in your proof.

4. Let (B_t) denote standard Brownian motion. Show that the following two processes have the same law:
   (a) The solution $(Y_t)_{0 \leq t \leq 1}$ of the SDE $\mathrm{d}Y_t = \frac{-Y_t}{1-t}\mathrm{d}t + \mathrm{d}B_t$ with $Y_0 = 0$.
   (b) The process $(Z_t)_{0 \leq t \leq 1}$, where $Z_t = B_t - tB_1$.