Stoch. Anal. Exam, January 8, 2019

Info: Each of the 4 questions is worth 12.5 marks. Write your name and Neptun code on each piece of paper that you submit. Separate the solutions of different exercises (and sub-exercises) with a horizontal line. No calculators or electronic devices are allowed. One hand-written two-sided A4-sized formula sheet with 30 formulas is allowed. You have 90 minutes to complete this exam.

1. Let us consider the filtration $\mathbb{F} = (\mathcal{F}_n)_{n=0}^{\infty}$ and assume that $\mathcal{F}_0$ is the trivial sigma-algebra.

   Assume that the stochastic process $(X_n)_{n=1}^{\infty}$ is adapted to $\mathbb{F}$ and that $\mathbb{E}(|X_n|) < +\infty$ for any $n = 1, 2, \dots$

   Prove that there exists a predictable process $(A_n)_{n=1}^{\infty}$ and a martingale $(M_n)_{n=1}^{\infty}$ satisfying $\mathbb{E}(M_n) = 0$ such that $X_n = A_n + M_n$ for all $n = 1, 2, \dots$ Prove that such a decomposition of $(X_n)_{n=1}^{\infty}$ is unique.

   Hint: $M_n = \sum_{i=1}^{n} (X_i - \mathbb{E}(X_i \,|\, \mathcal{F}_{i-1}))$ and $A_n = \mathbb{E}(X_1) + \sum_{i=2}^{n} (\mathbb{E}(X_i \,|\, \mathcal{F}_{i-1}) - X_{i-1})$

2. Let $f : \mathbb{R} \to \mathbb{R}$ denote a twice continuously differentiable function and let us assume that $M := \max_{0 \le s \le t} |f''(s)| < +\infty$. Assume given a partition $\Delta = \{t_0, t_1, \dots, t_n\}$ of the interval $[0, t]$. Let $(B(t))$ denote standard Brownian motion. Let

$$\mathcal{J} := \frac{1}{2} \sum_{k=1}^{n} f''(B(t_{k-1}))(B(t_k) - B(t_{k-1}))^2, \qquad \mathcal{L} := \frac{1}{2} \sum_{k=1}^{n} f''(B(t_{k-1}))(t_k - t_{k-1}).$$

   Prove that $\mathbb{E}[(\mathcal{J} - \mathcal{L})^2] \le \frac{M^2}{2} \sum_{k=1}^{n} (t_k - t_{k-1})^2$.

   Hint: Introduce an auxiliary discrete-time martingale $N_1, \dots, N_n$ and apply the Pythagorean theorem for matringales. Also good to know: if $Z \sim \mathcal{N}(0, \sigma^2)$ and $X = Z^2$ then $\mathrm{Var}(X) = 2\sigma^4$.

3. *Log-optimal portfolio.* You are trading shares at the stock market. The value of one stock at time t is

$$S_t = \sigma B_t + \mu t,$$

   where (B_t) is standard Brownian motion and $\mu > 0$ (this is a toy example so let's not worry about the fact that S_t can become negative). At time t you hold C_t shares. It is OK to go in debt (i.e., it is OK if $C_t > Y_t$). Denote by $Y_0 > 0$ your initial wealth and by Y_t your total wealth at time t. Let us assume that you can't predict the future, so that (C_t) is left-continuous and adapted to $(\mathcal{F}_t)$, where $\mathcal{F}_t = \sigma(S_u,\, 0 \le u \le t)$. Your net gain from trading shares at time t is

$$Y_t - Y_0 = \int_0^t C_u \, \mathrm{d}S_u.$$

   You trade in stocks until time T. Your goal is to maximize your expected rate of return $\mathbb{E}\left[\log\left(\frac{Y_T}{Y_0}\right)\right]$. What is the maximal expected rate of return achievable and the trading strategy that achieves it?

   Hint: First calculate $\mathrm{d}\log(Y_t)$ and express the result using the notation $X_t := C_t / Y_t$. Then you want to choose X_t in a way that the drift of $\mathrm{d}\log(Y_t)$ is maxized.

4. (a) (8 marks) Find the strong solution of the SDE

$$\mathrm{d}X_t = -2X_t \, \mathrm{d}t + \mathrm{d}B_t, \qquad X_0 = 1.$$

   (b) (4.5 marks) What is the distribution of X_1?