Stoch. Anal. Exam, June 27, 2023

Info: Each of the 4 questions is worth 12.5 marks. Write your name and Neptun code on each piece of paper that you submit. Separate the solutions of different exercises (and sub-exercises) with a horizontal line. No calculators or electronic devices are allowed. One hand-written two-sided A4-sized formula sheet with 30 formulas is allowed. You have 90 minutes to complete this exam.

1. Let (B_t) denote a standard Brownian motion. Let $M_1 := \max_{0 \le t \le 1} B_t$.
   (a) (4 points) Let $X_t := B(1) - B(1-t)$ for any $0 \le t \le 1$. Show that $(X_t)_{0 \le t \le 1}$ is also a standard Brownian motion on the time interval $[0,1]$.
   (b) (4.5 points) Use the reflection principle to show that M_1 has the same distribution as $|B_1|$.
   (c) (4 points) Show that M_1 has the same distribution as $M_1 - B_1$.

2. *Martingale representation theorem for simple random walk.* Let $S_n = \xi_1 + \cdots + \xi_n$, where $\xi_1, \xi_2, \ldots,$ are i.i.d. and $\mathbb{P}(\xi_k = 1) = \mathbb{P}(\xi_k = -1) = \frac{1}{2}$, $k \ge 1$. Let $(\mathcal{F}_n)_{n \ge 0}$ denote the natural filtration of the simple random walk (S_n), i.e., $\mathcal{F}_n = \sigma(S_1, \ldots, S_n) = \sigma(\xi_1, \ldots, \xi_n)$.
   (a) (7 points) Show that any martingale (M_n) with $M_0 = 0$ adapted to $(\mathcal{F}_n)_{n \ge 0}$ is a discrete stochastic integral $(H \cdot S)_n$ of a predictable process (H_n) with respect to the martingale (S_n). Explicitly state the formula for H_n. *Hint:* Since M_n is $\sigma(\xi_1, \ldots, \xi_n)$-measurable, there exists a function $\varphi_n : \{-1,1\}^n \to \mathbb{R}$ such that $M_n = \varphi_n(\xi_1, \ldots, \xi_n)$. Please use this notation in your proof.
   (b) (5.5 points) Show that if X is an $\mathcal{F}_{100}$-measurable random variable then
   $$X = \mathbb{E}[X] + \sum_{k=1}^{100} Y_k \cdot (S_k - S_{k-1})$$
   for some random variables $Y_1, \ldots, Y_{100}$, where Y_k is $\mathcal{F}_{k-1}$-measurable, $k = 1, \ldots, 100$.

3. Let $(\underline{B}_t)$ denote 7-dimensional Brownian motion. Show that if $Z_t = \|\underline{B}_t\|$, then (Z_t) is a weak solution of the SDE
   $$\mathrm{d}Z_t = \frac{3}{Z_t}\mathrm{d}t + \mathrm{d}\widetilde{B}_t.$$
   Instruction: Explain why the process $(\widetilde{B}_t)$ that appears in your solution is a Brownian motion.

   Instruction2: Explain briefly why (Z_t) is a weak (i.e., not strong) solution of the above SDE.

4. Let us assume that the price of a stock at time t is S_t, where $\mathrm{d}S_t = mS_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}B_t$. Let us assume that the price of a bond at time t is R_t, where $\mathrm{d}R_t = rR_t\mathrm{d}t$. Let us denote by $f(t,x)$ the fair price at time $t \in [0,T]$ of a European call option with strike price K and maturity date T under the assumption that $S_t = x$, where $t \in [0,T]$ and $x \in \mathbb{R}_+$. Use hedging to derive that
   $$\partial_t f(t,x) = rf(t,x) - rx\partial_x f(t,x) - \frac{\sigma^2 x^2}{2}\partial^2_{xx} f(t,x), \qquad f(T,x) = (x-K)_+.$$
   Also describe the hedging strategy that you used to replicate the option.