Stoch. Anal. Exam, 13.01.2015

Info: Each of the 4 questions is worth 12.5 marks. Write your name and Neptun code on each piece of paper that you submit. Separate the solutions of different exercises (and sub-exercises) with a horizontal line. No calculators or electronic devices are allowed. One hand-written A4-sized formula sheet is allowed. You have 110 minutes to complete this exam.

1. You are in a casino. If you bet 1 dollar in the n'th round then your your net profit in that round is ξ_n dollars where $\mathbb{P}(\xi_n = +1) = p$, $\mathbb{P}(\xi_n = -1) = q$, $q + p = 1$, $p > 1/2$ and $\xi_1, \xi_2, \dots$ are i.i.d. In other words: with probability $q < 1/2$ you lose the bet and with probability $p = 1 - q > 1/2$ you double your bet. You bet C_n dollars in the n'th round. Denote by Y_0 your initial wealth and by Y_n your total wealth after the end of the n'th round. You cannot go in debt, so let's assume $0 \le C_n \le Y_{n-1}$, $n > 0$. You are allowed to play N rounds. Your goal is to maximize your expected rate of return $\mathbb{E}(\log Y_N - \log Y_0)$. Denote by $\mathcal{F}_n = \sigma(\xi_1, \dots, \xi_n)$.

   (a) Prove that for any predictable betting strategy C_n the process $Z_n := n\alpha - \log Y_n$ is a submartingale, where $\alpha = p \log p + q \log q + \log 2$. Show that this implies $\mathbb{E}(\log Y_N - \log Y_0) \le N\alpha$.

   (b) Show that there is a betting strategy for which Z_n is a martingale and that $\mathbb{E}(\log Y_N - \log Y_0) = N\alpha$ is achieved. (Sometimes they call this the *log-optimal portfolio* in economics)

   Help: If the function $f(x) = p \ln(1+x) + (1-p)\ln(1-x)$ is defined for $0 \le x \le 1$, then $f'(x) = \frac{p}{1+x} - \frac{1-p}{1-x}$ and $f''(x) = \frac{-p}{(1+x)^2} - \frac{1-p}{(1-x)^2}$.

2. *Polarization:* $\frac{(a+b)^2 - a^2 - b^2}{2} = ab$. Let $(X_1(t))_{0 \le t \le T}$ and $(X_2(t))_{0 \le t \le T}$ be left-continuous adapted stochastic processes satisfying $\mathbb{E}\left(\int_0^T X_i^2(t)\,\mathrm{d}t\right) < +\infty$ for $i = 1, 2$, and denote by $Y_i(t) = \int_0^t X_i(s)\mathrm{d}B_s$, $i = 1, 2$ the corresponding Itō integrals with respect to the same Brownian motion (B_t).

   (a) Use the identity $\mathbf{Var}\left(\int_0^T X_t\,\mathrm{d}B_t\right) = \mathbb{E}\left(\int_0^T X_t^2\,\mathrm{d}t\right)$ to calculate $\mathbf{Cov}(Y_1(T), Y_2(T))$.

   (b) Use the fact that $M_t = \left(\int_0^t X_s\,\mathrm{d}B_s\right)^2 - \int_0^t X_s^2\,\mathrm{d}s$ is a martingale to find the the process (Ψ_t) such that $N_t = Y_1(t)Y_2(t) - \int_0^t \Psi_s\,\mathrm{d}s$ is a martingale.

   (c) Use that fact that the quadratic variation $[Y]_t$ of the process $Y_t = \int_0^t X_s\,\mathrm{d}B_s$ is $[Y]_t = \int_0^t X_s^2\,\mathrm{d}s$ to calculate the mutual variation $[Y_1, Y_2]_t$.

3. Find the explicit solution of the SDE

$$\mathrm{d}X_t = \frac{X_t + 1}{t - 1}\mathrm{d}t + \mathrm{d}B_t, \qquad 0 \le t < 1, \quad X_0 = 1.$$

4. Let $X_t^* = x_0 + \int_0^t \mu(X_s)\,\mathrm{d}s + \int_0^t \sigma(X_s)\,\mathrm{d}B_s$, where $\mu(\cdot)$ and $\sigma(\cdot)$ are Lipschitz-continuous. Let $T \in \mathbb{R}_+$. Let $X_0(t) \equiv x_0 \in \mathbb{R}$ and $X_n(t) := X_{n-1}^*(t)$, $n \ge 1$. Show that there exists a continuous process $(X_\infty(t))_{0 \le t \le T}$ such that $(X_n(t))$ converges uniformly on $[0, T]$ to $(X_\infty(t))$ $\mathbb{P}$-almost surely as $n \to \infty$.

   In your solution you can use the following facts without proving them, however you are asked to clearly indicate where you used them:

   (i) $\mathbb{E}\left[\int_0^T (X_{n+1}(t) - X_n(t))^2\,\mathrm{d}t\right] \le c_n^2$, where $\sum_{n=1}^\infty \sqrt{c_n} < +\infty$.

   (ii) Itō isometry; Markov inequality; submartingale inequality and its corollaries; the uniform limit of continuous functions is a continuous function; Borel-Cantelli lemma.