Stoch. Anal. Exam, January 10, 2017

Info: Each of the 4 questions is worth 12.5 marks. Write your name and Neptun code on each piece of paper that you submit. Separate the solutions of different exercises (and sub-exercises) with a horizontal line. No calculators or electronic devices are allowed. One hand-written two-sided A4-sized formula sheet with 30 formulas is allowed. You have 100 minutes to complete this exam.

1. Let (B_t) denote standard Brownian motion. Let us fix $T \in \mathbb{R}_+$. Given the partition

$$\Delta_n = \{t_0, t_1, \ldots, t_n\}, \qquad t_0 < t_1 < \cdots < t_n, \qquad t_0 = 0, \quad t_n = T,$$

let us define

$$Q_n = \sum_{k=1}^{n} \big(B(t_k) - B(t_{k-1})\big)^2.$$

We assume that the sequence of partitions (Δ_n) satisfies $|\Delta_n| \to 0$, where $|\Delta_n| = \max_{1\le k\le n}(t_k - t_{k-1})$.

   (a) (4 marks) Calculate $\mathbb{E}(Q_n)$, $\mathrm{Var}(Q_n)$. *Hint:* If $X \sim \mathcal{N}(0, \sigma^2)$ then $\mathrm{Var}(X^2) = 2\sigma^4$.

   (b) (3 marks) Calculate the quadratic variation of (B_t) on $[0, T]$. More precisely, show that $[B]_T = T$. *Hint:* You have to show that $Q_n \xrightarrow{\mathbb{P}} T$ as $|\Delta_n| \to 0$. Use part (a) and *Chebyshev's inequality*.

   (c) (5.5 marks) Let us define

$$I_n = \sum_{k=1}^{n} B(t_{k-1}) \cdot (B(t_k) - B(t_{k-1})).$$

   Find the limit of I_n (in probability) as $|\Delta_n| \to 0$.

2. Prove that if $f : \mathbb{R} \to \mathbb{R}$ and $\max_{x\in\mathbb{R}} |f''(x)| < +\infty$ and (B_t) is standard Brownian motion, then

$$f(B_t) - f(B_0) = \int_0^t f'(B_s)\mathrm{d}B_s + \frac{1}{2}\int_0^t f''(B_s)\mathrm{d}s.$$

Instruction: The starting point of your proof should be the Taylor formula with remainder term. In order to make your proof shorter, you are allowed to use lemmas without proving them, but you have to precisely state those lemmas and clearly indicate where you used them in your proof.

3. (a) (2 marks) Define the notion of a *stationary* stochastic process (in the continuous-time setting).

   (b) (7 marks) Let us define the stochastic process $(Y_t)_{t\ge 0}$ by

$$Y_t = e^{-t}Y_0 + \sqrt{2}\int_0^t e^{u-t}\,\mathrm{d}B_u,$$

   where $Y_0 \sim \mathcal{N}(0, 1)$, moreover (B_t) is standard Brownian motion, and Y_0 is independent from (B_t). Calculate $\mathbb{E}(Y_t)$ and $\mathrm{Cov}(Y_s, Y_t)$ for any $s, t \ge 0$.

   (c) (3.5 marks) Prove that (Y_t) is a *stationary* process.
   Instruction: It is allowed to use that (Y_t) is a *Gaussian* process without proving it.

4. Let (X_t) be a Brownian motion with constant (positive) drift, i.e., the solution of the SDE

$$\mathrm{d}X_t = \mu\mathrm{d}t + \sigma\mathrm{d}B_t, \qquad X_0 = x_0 \in \mathbb{R}, \qquad \mu > 0,\ \sigma > 0.$$

   (a) (8 marks) Find a non-constant function $f : \mathbb{R} \to \mathbb{R}$ such that $M_t = f(X_t)$ is a martingale.

   (b) (4.5 marks) Use (M_t) and the optional stopping theorem to calculate $\mathbb{P}(T_a < T_b)$ where

$$a \le x_0 \le b, \qquad T_a = \min\{\, t \,:\, X_t = a \,\}, \qquad T_b = \min\{\, t \,:\, X_t = b \,\}$$

   Hint: This exercise is the continuous analogue of the *gambler's ruin problem* (HW 3.1), which was about a biased lazy random walk (rather than a Brownian motion with constant drift).
   Instruction: You don't have to check that the optional stopping theorem can be applied here.