Stoch. Anal. Exam, January 15, 2019

Info: Each of the 4 questions is worth 12.5 marks. Write your name and Neptun code on each piece of paper that you submit. Separate the solutions of different exercises (and sub-exercises) with a horizontal line. No calculators or electronic devices are allowed. One hand-written two-sided A4-sized formula sheet with 30 formulas is allowed. You have 90 minutes to complete this exam.

1. (a) (2 marks) Define the mutual variation $[X,Y]_t$ of the stochastic processes (X_t) and (Y_t).
   (b) (3 marks) State and prove the polarization identity for mutual variations.
   Instruction: You may use without proof the bilinearity of mutual variation.
   (c) (4 marks) Show that if (B_t) and (B_t^*) are independent standard Brownian motions, then $(\widetilde{B}_t)$ is also a Brownian motion, where $\widetilde{B}_t = \frac{1}{\sqrt{2}}(B_t + B_t^*)$.
   (d) (3.5 marks) Calculate the mutual variation $[B, B^*]_t$.
   Instruction: You may use without proof that $[B]_t = t$.

2. *Discrete Itô formula.* We have learnt Itô's formula is class: $f(B_t) - f(B_0) = \int_0^t f'(B_s)\,\mathrm{d}B_s + \frac{1}{2}\int_0^t f''(B_s)\,\mathrm{d}s$. The goal of this exercise is to find the discrete analogue of this formula.

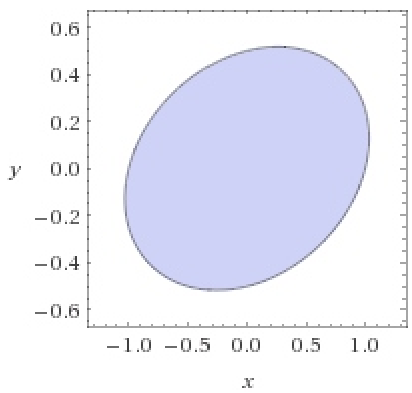
   Let $S_n = \xi_1 + \cdots + \xi_n$, where $\xi_1, \xi_2, \ldots,$ are i.i.d. and $\mathbb{P}(\xi_k = 1) = \mathbb{P}(\xi_k = -1) = \frac{1}{2}$, $k \geq 1$. Let $(\mathcal{F}_n)_{n\geq 0}$ denote the natural filtration of the simple random walk (S_n), i.e., $\mathcal{F}_n = \sigma(S_1, \ldots, S_n) = \sigma(\xi_1, \ldots, \xi_n)$.

   Given some $f : \mathbb{Z} \to \mathbb{R}$ let us define $f^*(x) := \frac{f(x+1)-f(x-1)}{2}$ and $f^{**}(x) := f(x+1) - 2f(x) + f(x-1)$.

   (a) (6 marks) Let us define $X_n = f(S_n) - f(S_0)$. Find the discrete Doob-Meyer decomposition of the process (X_n), i.e., write $X_n = A_n + M_n$, where (A_n) is a predictable process and (M_n) is a martingale with zero expectation. Explicitly state the formula for A_n using the function $f^{**}(\cdot)$.
   (b) (6 marks) Write M_n as the discrete stochastic integral $(H \cdot S)_n$ of a predictable process (H_n) with respect to the martingale (S_n). Explicitly state the formula for H_n using the function $f^*(\cdot)$.
   (c) (0.5 mark) Put the results of (a) and (b) together to obtain a formula for $f(S_n) - f(S_0)$ that looks like the discrete version of the right-hand side of Itô's formula.

3. Let $\underline{B}_t = (B_1(t), B_2(t))$ denote standard 2-dimensional Brownian motion.

   (a) (6 marks) Find the constant α for which $M_t = B_1(t)^2 - B_1(t)B_2(t) + 4B_2(t)^2 - \alpha t$ is a martingale.
   (b) (6.5 marks) We define the planar region $\mathcal{E} = \{\,(x,y) \;:\; x^2 - xy + 4y^2 \leq 1\,\}$, which is an ellipse centered at the origin, see figure below. Denote by $\tau = \inf\{\, t : \underline{B}_t \notin \mathcal{E}\,\}$ the first time the Brownian motion $(\underline{B}_t)$ exits the region $\mathcal{E}$. Calculate the expected value of τ using the optional stopping theorem. *Instruction:* You do not need to check the conditions of the optional stopping theorem.



4. Calculate the solution (V_t) of integral equation $V_t = v_0 + \int_0^t V_s\,\mathrm{d}X_s + Y_t - Y_0$ given the Itō processes (X_t) and (Y_t). *Instruction:* You may assume without proof that the solution of the equation $U_t = 1 + \int_0^t U_s\,\mathrm{d}X_s$ is $U_t = \exp\left(X_t - X_0 - \frac{1}{2}[X]_t\right)$. *Hint:* The result should be $V_t = U_t \cdot \left(v_0 + \int_0^t \frac{1}{U_s}\,\mathrm{d}Y_s - \int_0^t \frac{1}{U_s}\,\mathrm{d}[X,Y]_s\right)$.