Stoch. Anal. Exam, July 6, 2023

Info: Each of the 4 questions is worth 12.5 marks. Write your name and Neptun code on each piece of paper that you submit. Separate the solutions of different exercises (and sub-exercises) with a horizontal line. No calculators or electronic devices are allowed. One hand-written two-sided A4-sized formula sheet with 30 formulas is allowed. You have 90 minutes to complete this exam.

1. Let (X, Y) denote a pair of continuous random variables with joint density function

$$f(x, y) = c \exp\left(-\frac{5}{2}x^2 - 3xy - y^2\right).$$

   Instruction: instead of calculating integrals, solve the following sub-exercises by using the properties of multivariate normal distribution.

   (a) (3 points) Find the covariance matrix of (X, Y).

   (b) (3 points) Find c.

   (c) (3 points) Find the density function $f_X(x)$ of X.

   (d) (3.5 points) Find $\mathbb{E}(\, Y \mid \sigma(X)\,)$. *Hint:* First try to find $\lambda \in \mathbb{R}$ such that $Z := \lambda \cdot X + Y$ is independent of X.

2. (5+5+2.5 points) Let $S_n = \xi_1 + \cdots + \xi_n$, where $\xi_1, \xi_2, \ldots,$ are i.i.d. and

$$\mathbb{P}(\xi_k = 1) = \mathbb{P}(\xi_k = -1) = \frac{1}{2}, \qquad k \geq 1.$$

   Let $(\mathcal{F}_n)_{n\geq 0}$ denote the natural filtration of the simple random walk (S_n), i.e., $\mathcal{F}_n = \sigma(S_1, \ldots, S_n) = \sigma(\xi_1, \ldots, \xi_n)$. Given some $f : \mathbb{Z} \to \mathbb{R}$ let us define

$$f^*(x) := \frac{f(x+1) - f(x-1)}{2}, \qquad f^{**}(x) := f(x+1) - 2f(x) + f(x-1)$$

   (a) Let us define $X_n = f(S_n)$. Find the discrete Doob-Meyer decomposition of the process (X_n), i.e., write $X_n = A_n + M_n$, where (A_n) is a predictable process and (M_n) is a martingale with zero expectation. State the formula for A_n using the function $f^{**}(\cdot)$.

   (b) Write M_n as the discrete stochastic integral $(H \cdot S)_n$ of a predictable process (H_n) with respect to the martingale (S_n). State the formula for H_n using the function $f^*(\cdot)$.

   (c) Put the results of (a) and (b) together to obtain a formula for $f(S_n) - f(S_0)$ that looks like the discrete version of the right-hand side of Itô's formula.

3. (a) (6.5 points) Given an Itō process (Y_t), find the Itō process (Z_t) for which

$$Z_t = 1 + \int_0^t Z_s \, \mathrm{d}Y_s, \qquad t \geq 0. \tag{1}$$

   Instruction: The final solution formula for (Z_t) can contain Stieltjes integrals, but it should not contain Itō integrals. Please provide the details of your calculations using Itō's formula.

   (b) (6 points) Given an Itō process (Z_t) and Y_0, find the Itō process (Y_t) for which (1) holds.
   Instruction: The final solution formula for (Y_t) can contain Stieltjes integrals, but it should not contain Itō integrals. Please provide the details of your calculations using Itō's formula.

4. (a) (4 points) State Girsanov's theorem.

   (b) (8.5 points) Prove Girsanov's theorem.

   Instruction: You can use without proof that if (u_t) is left-continuous, adapted, and Novikov's condition holds and if we define

$$X_t := B_t + \int_0^t u_s \, \mathrm{d}s, \qquad M_t := \exp\left(\int_0^t (-u_s) \, \mathrm{d}B_s - \frac{1}{2}\int_0^t u_s^2 \, \mathrm{d}s\right), \qquad Y_t := X_t M_t,$$

   then (M_t) and (Y_t) are both martingales.