Stoch. Anal. Exam, June 18, 2025

Info: Write your name and Neptun code on each piece of paper that you submit. Separate the solutions of different exercises (and sub-exercises) with a horizontal line. No calculators or electronic devices are allowed. One hand-written two-sided A4-sized formula sheet with 30 formulas is allowed. You have 90 minutes to complete this exam.

1. Let us assume that the function $f : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable and that $\|f''\|_\infty := \sup_{x\in\mathbb{R}} |f''(x)| < +\infty$. Let $0 = t_0 < t_1 < \cdots < t_n = t$ and let

$$\mathcal{I}_n := \sum_{k=1}^{n} f'(B(t_{k-1})) \cdot (B(t_k) - B(t_{k-1})).$$

   (a) (4 points) Write down the simple predictable process $(X_n(s))_{0\le s\le t}$ for which $\mathcal{I}_n = \int_0^t X_n(s)\,\mathrm{d}B_s$.

   (b) (8 points) Show that

$$\mathbb{E}\left[\left(\mathcal{I}_n - \int_0^t f'(B_s)\,\mathrm{d}B_s\right)^2\right] \le \frac{1}{2}\|f''\|_\infty^2 \sum_{k=1}^{n}(t_k - t_{k-1})^2.$$

2. (12 points) Let $(\underline{B}_t)$ denote n-dimensional Brownian motion. Show that if $Y_t = \|\underline{B}_t\|$, then (Y_t) is a weak solution of the SDE

$$\mathrm{d}Y_t = \frac{n-1}{2}\frac{1}{Y_t}\mathrm{d}t + \mathrm{d}\widetilde{B}_t.$$

   Instruction: explain where you used Paul Lévy's characterization of Brownian motion in your proof.

3. (a) (6 points) Write down the SDE of geometric Brownian motion and solve the SDE.

   (b) (7 points) State Kolmogorov's backward equation (you don't need to prove it) and use it to show that the solution of the PDE

$$\partial_t u(t,x) = \alpha x \partial_x u(t,x) + \frac{1}{2}\beta^2 x^2 \partial_{xx} u(t,x), \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}_+, \tag{1}$$

$$u(0,x) = f(x), \qquad x \in \mathbb{R} \tag{2}$$

   is equal to

$$u(t,x) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} f\big(x \exp\{\beta y + (\alpha - \beta^2/2)t\}\big) e^{-\frac{y^2}{2t}}\,\mathrm{d}y, \qquad t, x > 0.$$

4. Let $\widetilde{B}_t = B_t + \mu \cdot t$ (i.e., let $(\widetilde{B}_t)$ denote a Brownian motion with constant drift that starts from $\widetilde{B}_0 = 0$). Let

$$M_1 := \max_{0\le t\le 1} B_t, \qquad \widetilde{M}_1 := \max_{0\le t\le 1} \widetilde{B}_t.$$

   Let $f_0(x,y)$ denote the joint p.d.f. of (M_1, B_1). Let $f_\mu(x,y)$ denote the joint p.d.f. of $(\widetilde{M}_1, \widetilde{B}_1)$.

   (a) (6 points) Express $f_\mu(x,y)$ in terms of $f_0(x,y)$ using the Cameron-Martin-Girsanov theorem.

   (b) (7 points) Find $f_0(x,y)$. *Hint:* First find the joint c.d.f. $F_0(x,y) = \mathbb{P}(M_1 \le x,\, B_1 \le y)$ using the reflection principle.