Stoch. Anal. Exam, 20.01.2015

Info: Each of the 4 questions is worth 12.5 marks. Write your name and Neptun code on each piece of paper that you submit. Separate the solutions of different exercises (and sub-exercises) with a horizontal line. No calculators or electronic devices are allowed. One hand-written A4-sized formula sheet is allowed. You have 110 minutes to complete this exam.

1. For each $n \in \mathbb{N}$, let $\Delta_n = \{t_0^n, t_1^n, \dots, t_n^n\}$ denote a partition of the interval $[0,1]$, i.e.,

$$0 = t_0^n < t_1^n < \cdots < t_n^n = 1.$$

Let $(B(t))$ denote standard Brownian motion. Let us define

$$|\Delta_n| = \max_{0 \le k \le n-1} |t_{k+1}^n - t_k^n|$$

and let us assume that $\lim_{n\to\infty} |\Delta_n| = 0$. Calculate

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} B(t_k^n) \cdot \left(B(t_{k+1}^n) - B(t_k^n) \right),$$

where the limit is in the sense of convergence in probability of a sequence of random variables.

Note: In your solution you can use what we have learnt in class about the quadratic variation of $(B(t))$.

2. Let $(B_1(t))$ and $(B_2(t))$ denote independent standard one-dimensional Brownian motions starting from 0 and let $\underline{B}(t) = (B_1(t), B_2(t))$ be a standard 2-dimensional Brownian motion. For any $\underline{x} = (x, y) \in \mathbb{R}^2$ let $\|\underline{x}\| = \sqrt{x^2 + y^2}$ denote the Euclidean norm of $\underline{x}$. Let $\tau = \min\{\, t : \|\underline{B}(t)\| = 1 \,\}$.

   (a) Let $h : \mathbb{R}^2 \to \mathbb{R}$ be a harmonic function (i.e., $\Delta h \equiv 0$). Prove

$$h(0,0) = \frac{1}{2\pi} \int_0^{2\pi} h\left(\cos(\varphi),\, \sin(\varphi)\right) \mathrm{d}\varphi.$$

   (b) Calculate $\mathbb{E}(\tau)$.

   (c) Calculate $\mathbb{E}\left[f(\underline{B}(\tau)) \right]$, where $f(x,y) = (x+1)^3 - 3xy^2$.

   Note: You can use the optional stopping theorem without checking the uniform integrability condition.

3. (a) Solve the homogeneous linear equation $\mathrm{d}X_t = B_t X_t \mathrm{d}t + B_t X_t \mathrm{d}B_t$ with initial condition $X_0 = 1$.
   Hint: First calculate $\mathrm{d}\ln(X_t)$.

   (b) Solve the inhomogeneous linear equation $\mathrm{d}Y_t = B_t Y_t \mathrm{d}t + B_t Y_t \mathrm{d}B_t + B_t \mathrm{d}B_t$ with $Y_0 = 1$.
   Hint: Variation of constants: $Y_t = X_t C_t$.

4. Prove that if (X_t) and (Y_t) are both adapted and continuous stochastic processes on $[0,T]$ satisfying

$$\mathbb{E}\left[\int_0^T X_t^2 \, \mathrm{d}t \right] < +\infty \qquad \text{and} \qquad \mathbb{E}\left[\int_0^T Y_t^2 \, \mathrm{d}t \right] < +\infty,$$

moreover they both satisfy the same SDE with the same initial condition

$$X_t = x_0 + \int_0^t \mu(X_s)\, \mathrm{d}s + \int_0^t \sigma(X_s)\, \mathrm{d}B_s, \qquad 0 \le t \le T$$

$$Y_t = x_0 + \int_0^t \mu(Y_s)\, \mathrm{d}s + \int_0^t \sigma(Y_s)\, \mathrm{d}B_s, \qquad 0 \le t \le T$$

where $\mu : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}$ are both Lipschitz-continuous functions, then

$$\mathbb{P}\left[\, X_t \equiv Y_t,\ 0 \le t \le T \,\right] = 1.$$

Note: In your solution you can use the following facts without proving them, however you are asked to clearly indicate where you used them: Itō isometry; Cauchy-Schwarz inequality; Grönwall's inequality.