Stoch. Anal. Exam, January 17, 2017

Info: Each of the 4 questions is worth 12.5 marks. Write your name and Neptun code on each piece of paper that you submit. Separate the solutions of different exercises (and sub-exercises) with a horizontal line. No calculators or electronic devices are allowed. One hand-written two-sided A4-sized formula sheet with 30 formulas is allowed. You have 100 minutes to complete this exam.

1. *The Stieltjes integral is well-defined.* Let $f : [0,t] \to \mathbb{R}$ be a function of finite total variation, i.e., $V_f(t) < +\infty$. Let $g : [0,t] \to \mathbb{R}$ be a continuous function.

   (a) Show that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any partitions $\Delta = \{t_0 < \cdots < t_n\}$ and $\tilde{\Delta} = \{\tilde{t}_0 < \cdots < \tilde{t}_n\}$ of $[0,t]$ satisfying $|\Delta| \le \delta$ and $|\tilde{\Delta}| \le \delta$ and for any choice of sample points $\underline{t}^* = \{t_1^* \le \cdots \le t_n^*\}$ and $\underline{\tilde{t}}^* = \{\tilde{t}_1^* \le \cdots \le \tilde{t}_n^*\}$ satisfying $t_{k-1} \le t_k^* \le t_k$ and $\tilde{t}_{k-1} \le \tilde{t}_k^* \le \tilde{t}_k$ we have

   $$|\mathcal{I}(\Delta, \underline{t}^*) - \mathcal{I}(\tilde{\Delta}, \underline{\tilde{t}}^*)| \le \varepsilon, \qquad \text{where} \qquad \mathcal{I}(\Delta, \underline{t}^*) = \sum_{k=1}^{n} g(t_k^*) \cdot (f(t_k) - f(t_{k-1})) \,.$$

   Hint: Consider the partition $\Delta \cup \tilde{\Delta}$ which is a subdivision of both of the partitions Δ and $\tilde{\Delta}$.

   (b) Show that $\lim_{|\Delta_n| \to 0} \mathcal{I}(\Delta_n, \underline{t}_n^*)$ exists and is the same for any sequence (Δ_n) of partitions and sample points $(\underline{t}_n^*)$ (as long as $|\Delta_n| \to 0$).

2. Let $S_n = \xi_1 + \cdots + \xi_n$, where $\xi_1, \xi_2, \ldots,$ are i.i.d. and $\mathbb{P}(\xi_k = 1) = \mathbb{P}(\xi_k = -1) = \frac{1}{2}$, $k \ge 1$. Let $(\mathcal{F}_n)_{n \ge 0}$ denote the natural filtration of the simple random walk (S_n), i.e., $\mathcal{F}_n = \sigma(S_1, \ldots, S_n) = \sigma(\xi_1, \ldots, \xi_n)$.

   Given some $f : \mathbb{Z} \to \mathbb{R}$ let us define

   $$f^*(x) := \frac{f(x+1) - f(x-1)}{2}, \qquad f^{**}(x) := f(x+1) - 2f(x) + f(x-1)$$

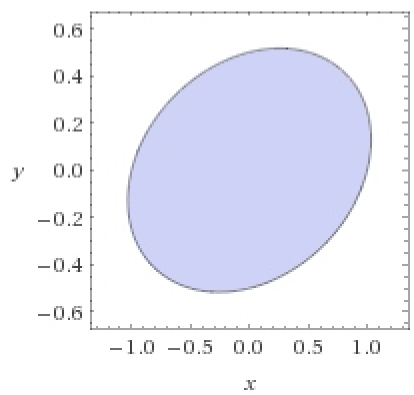
   (a) Let us define $X_n = f(S_n)$. Find the discrete Doob-Meyer decomposition of the process (X_n), i.e., write $X_n = A_n + M_n$, where (A_n) is a predictable process and (M_n) is a martingale with zero expectation. Explicitly state the formula for A_n using the function $f^{**}(\cdot)$.

   (b) Write M_n as the discrete stochastic integral $(H \cdot S)_n$ of a predictable process (H_n) with respect to the martingale (S_n). Explicitly state the formula for H_n using the function $f^*(\cdot)$.

   (c) Put the results of (a) and (b) together to obtain a formula for $f(S_n) - f(S_0)$ that looks like the discrete version of the right-hand side of the integral form of Itô's formula.

3. Let $\underline{B}_t = (B_1(t), B_2(t))$ denote standard 2-dimensional Brownian motion. We also define the planar region $\mathcal{E} = \{\, (x,y) \;:\; x^2 - xy + 4y^2 \le 1 \,\}$, which is an ellipse centered at the origin, see figure below. Denote by $\tau = \inf\{\, t \;:\; \underline{B}_t \notin \mathcal{E} \,\}$ the first time the Brownian motion $(\underline{B}_t)$ exits the region $\mathcal{E}$. Calculate the expected value of τ using the optional stopping theorem applied to well-chosen martingale.

   Hint: If you have no idea, wave to me and I will give you a hint about the form of the martingale.



4. (a) Let $r, \sigma \in \mathbb{R}_+$ and $z_0 \in \mathbb{R}$. Find the Itô process (Z_t) satisfying $Z_t = z_0 - \int_0^t r Z_s \mathrm{d}s + \int_0^t \sigma \mathrm{d}B_s,\ t \ge 0$.

   Hint: First find the SDE corresponding to the integral equation satisfied by (Z_t).

   (b) Let us fix some $t \in \mathbb{R}_+$. What is the distribution of the random variable Z_t?