Stoch. Anal. Exam, January 22, 2019

Info: Each of the 4 questions is worth 12.5 marks. Write your name and Neptun code on each piece of paper that you submit. Separate the solutions of different exercises (and sub-exercises) with a horizontal line. No calculators or electronic devices are allowed. One hand-written two-sided A4-sized formula sheet with 30 formulas is allowed. You have 90 minutes to complete this exam.

1. Let us define $X_n = 2 + Y_1 + \cdots + Y_n$, where $Y_1, Y_2, \ldots$ are i.i.d. random variables with distribution

$$\mathbb{P}(Y_k = -1) = \mathbb{P}(Y_k = 0) = \mathbb{P}(Y_k = 1) = \frac{1}{3}.$$

Denote by $(\mathcal{F}_n)$ the filtration generated by the stochastic process (X_n).

(a) (6.5 marks) Find the discrete Doob-Meyer decomposition of the process (X_n^2), i.e., write

$$X_n^2 = A_n + M_n,$$

where (A_n) is a predictable process and (M_n) is a martingale with zero expectation. Give a simple and explicit formula for A_n.

(b) (6 marks) Given some $\beta \in \mathbb{R}$ find the constant $\gamma \in \mathbb{R}$ such that (M_n) is a martingale, where

$$M_n = \exp\left(\beta X_n - \gamma n\right).$$

2. Let $(B(t))$ denote standard Brownian motion. Show that the three processes below have the same law.

(a) $(B(t) - tB(1))_{0 \le t \le 1}$

(b) the conditional law of Brownian motion $(B(t))_{0 \le t \le 1}$ under the condition $B(1) = 0$

(c) $\left((1-t)\int_0^t \frac{1}{1-s}\,\mathrm{d}B_s\right)_{0 \le t \le 1}$

Instruction: It is OK to assume without proof that the three processes above are Gaussian. It is also OK to recall some details of Paul Lévy's construction of Brownian motion without proving them.

3. Let $\Phi(x)$ denote the c.d.f. of $\mathcal{N}(0,1)$.

(a) (8 marks) Show that $(M_t)_{0 \le t < 1}$ is a martingale, where $M_t = \Phi(B_t/\sqrt{1-t})$.

(b) (4.5 marks) Let us define the random variable Z by

$$Z = \begin{cases} 1, & \text{if } B_1 > 0 \\ 0, & \text{if } B_1 \le 0. \end{cases}$$

Let $\mathcal{F}_t = \sigma\left(B_s,\, 0 \le s \le t\right)$ denote the sigma-algebra generated by the Brownian motion up to time t. Find the constant C and the process $(X_t)_{0 \le t \le 1}$ adapted to the filtration $(\mathcal{F}_t)_{0 \le t \le 1}$ such that

$$Z = C + \int_0^1 X_s\,\mathrm{d}B_s.$$

4. We know that (X_t) is an Itō process with $X_0 = 1$. We also know that the process (U_t) given by

$$U_t = \exp\left(tB_t\right)$$

solves the integral equation

$$U_t = 1 + \int_0^t U_s\,\mathrm{d}X_s, \quad t \in [0, +\infty).$$

Given this information, find X_t. Simplify your solution as much as possible.