Stoch. Anal. Exam, June 25, 2025

Info: Write your name and Neptun code on each piece of paper that you submit. Separate the solutions of different exercises (and sub-exercises) with a horizontal line. No calculators or electronic devices are allowed. One hand-written two-sided A4-sized formula sheet with 30 formulas is allowed. You have 90 minutes to complete this exam.

1. This is an exercise on discrete-time stochastic processes. Let $(\mathcal{F}_n)_{n=0}^{\infty}$ denote a filtration.
   (a) (3 points) Define the following types of stochastic processes: adapted, predictable, martingale.
   (b) (9 points) State and prove the theorem about the existence and uniqueness of the discrete Doob-Meyer decomposition of an adapted stochastic process with finite expectation.

2. (a) (6 points) Use Itô calculus to show that

$$M_2(t) = B_t^2 - t, \qquad M_4(t) = B_t^4 - 6tB_t^2 + 3t^2$$

   are martingales. *Hint:* First calculate the stochastic differential of $(M_2(t))$ and $(M_4(t))$.
   (b) (6 points) Given $a \in \mathbb{R}_+$ let us define the stopping time

$$\tau = \min\{\, t \;:\; |B_t| = a \,\}.$$

   Use the optional stopping theorem to calculate $\mathbb{E}(\tau)$ and $\mathbb{E}(\tau^2)$.
   Instruction: You don't have to check that the optional stopping thm can be applied here.

3. Let $y_0, \alpha, \beta \in \mathbb{R}_+$. Let us consider the following SDE:

$$\mathrm{d}Y_t = \alpha \cdot (1 - Y_t) \cdot Y_t \,\mathrm{d}t + \beta \cdot Y_t \,\mathrm{d}B_t, \qquad Y_0 = y_0. \tag{1}$$

   Let $B_t^* = \beta \cdot B_t + (\alpha - \frac{1}{2}\beta^2) \cdot t$.
   (a) (5 points) Let $X_t := \ln(Y_t)$. Calculate $\mathrm{d}X_t$, rewrite it as an SDE for (X_t) and simplify the result using the notation $\mathrm{d}B_t^*$.
   (b) (5 points) Let either $V_t := X_t - B_t^*$ or $V_t := X_t + B_t^*$ (one of the two choices works, the other one doesn't: you need to decide which one works). Calculate $\mathrm{d}V_t$, rewrite it as a separable ODE for (V_t) and solve it.
   (c) (3 points) Find the strong solution of the SDE (1) above.

4. (a) (9 points) State and prove the formula for the stationary distribution of a time-homogeneous Itô diffusion process.
   (b) (2 points) Demonstrate a time-homogeneous Itô diffusion process which does not have a stationary distribution. Explain what goes wrong with the formula derived in sub-exercise (a).
   (c) (2 points) Demonstrate a time-homogeneous Itô diffusion process which has a stationary distribution and find it using the formula derived in sub-exercise (a).