Stoch. Anal. Exam, 27.01.2015

Info: Each of the 4 questions is worth 12.5 marks. Write your name and Neptun code on each piece of paper that you submit. Separate the solutions of different exercises (and sub-exercises) with a horizontal line. No calculators or electronic devices are allowed. One hand-written A4-sized formula sheet is allowed. You have 120 minutes to complete this exam.

1. We defined Brownian motion on the interval $[0,1]$ by
$$B(t) = t \cdot X + \sum_{i=0}^{\infty} \sum_{j=1}^{2^i} f_j^i(t) \cdot Y_j^i, \qquad t \in [0,1],$$
where
$$f_j^i(t) = \left(\frac{1}{\sqrt{2}}\right)^i \cdot f_1^0\left(t2^i - j + 1\right) \quad \text{and} \quad f_1^0(t) = \left(\frac{1}{2} - |t - \frac{1}{2}|\right) \vee 0,$$
moreover X and Y_j^i, $i \geq 0$, $1 \leq j \leq 2^i$ are i.i.d. with standard Normal distribution.
   (a) Draw the graph of the functions $f_1^0(t)$, $f_1^1(t)$, $f_2^1(t)$, $0 \leq t \leq 1$.
   (b) Let $Z(t) = t \cdot X + \sum_{i=0}^{1} \sum_{j=1}^{2^i} f_j^i(t) \cdot Y_j^i$. Calculate $\mathrm{Var}\,(Z(2/3))$.
   (c) Show that $(B(t))_{0 \leq t \leq 1}$ is a continuous function with probability one. In your solution you can use the following facts without proving them, however you are asked to clearly indicate where you used them: the uniform limit of continuous functions is a continuous function; Borel-Cantelli lemma.

2. If $\underline{x} = (x,y) \in \mathbb{R}^2$, denote by $\|\underline{x}\| = \sqrt{x^2 + y^2}$ the Euclidean norm of $\underline{x}$.
The aim of this exercise is to show that if $\underline{B}_t = (B_1(t), B_2(t))$ is a 2-dimensional Brownian motion starting from $\underline{x}_0 \neq \underline{0}$ and if $\tau_\varepsilon = \inf\{t \,:\, \|\underline{B}_t\| \leq \varepsilon\}$ then $\mathbb{P}(\tau_\varepsilon < +\infty) = 1$, i.e., $\underline{B}_t$ always hits the ball of radius ε around the origin.
   (a) Calculate the stochastic differential $\mathrm{d}M_t$ of
$$M_t = \ln\left(\|\underline{B}_t\|\right).$$
   (b) Use the optional stopping theorem to calculate $\mathbb{P}[T_a < T_b]$ for any $0 < a < \|\underline{x}_0\| < b < +\infty$.
   (c) Show that $\mathbb{P}(\tau_\varepsilon < +\infty) = 1$. How did you use the continuity of $(\underline{B}_t)$?

3. We know that (Y_t) is an Itō process with $Y_0 = 1$. We also now that the process (X_t) given by
$$X_t = \exp\left(tB_t\right)$$
solves the integral equation
$$X_t = 1 + \int_0^t X_s \,\mathrm{d}Y_s.$$
Given this information, find Y_t.

4. (a) Solve the SDE
$$\mathrm{d}X_t = X_t\,\mathrm{d}t + \mathrm{d}B_t, \qquad X_0 = x_0.$$
   (b) The solution (X_t) is a time-homogeneous Markov process: find its transition probability density function, i.e., find $p_t(x,y)$ for which we have
$$\mathbb{P}\left(X_{s+t} \in [y, y+\mathrm{d}y] \,|\, \mathcal{F}_s\right) = p_t(X_s, y)\,\mathrm{d}y, \qquad s,t \geq 0.$$