Stoch. Anal. Exam, January 24, 2017

Info: Each of the 4 questions is worth 12.5 marks. Write your name and Neptun code on each piece of paper that you submit. Separate the solutions of different exercises (and sub-exercises) with a horizontal line. No calculators or electronic devices are allowed. One hand-written two-sided A4-sized formula sheet with 30 formulas is allowed. You have 100 minutes to complete this exam.

1. Let $S_n = \xi_1 + \cdots + \xi_n$, where $\xi_1, \xi_2, \ldots,$ are i.i.d. and $\mathbb{P}(\xi_k = 1) = \mathbb{P}(\xi_k = -1) = \frac{1}{2}$, $k \geq 1$. Let $(\mathcal{F}_n)_{n\geq 0}$ denote the natural filtration of (S_n).

   (a) How to choose $C \in \mathbb{R}_+$ if we want $M_n = 2^{S_n}/C^n$ to be a martingale?

   (b) Write $M_n - M_0$ as the discrete stochastic integral $(H \cdot S)_n$ of a predictable process (H_n) with respect to the martingale (S_n). Explicitly state the formula for H_n.

2. *Equivalent definitions of the Brownian bridge.* Denote by $(B(t))$ the standard Brownian motion. Show that the three processes below (each one is defined for $0 \leq t \leq 1$) are Gaussian and that the three processes have the same law.

   (a) $B(t) - tB(1)$.

   (b) $(1-t)B(t/(1-t))$.

   (c) $(1-t)\int_0^t \frac{1}{1-s}\, \mathrm{d}B_s$

3. (a) Find the explicit solution of the stochastic differential equation

   $$\mathrm{d}Y_t = aY_t\mathrm{d}t + bY_t\mathrm{d}B_t, \qquad Y_0 = c, \qquad a, b, c \in \mathbb{R}_+$$

   (b) Find an explicit formula for the cumulative distribution function $F(x) = \mathbb{P}(Y_2 \leq x)$, $x \in \mathbb{R}$ of the random variable Y_2.

   Hint: You may use $\Phi(\cdot)$, the c.d.f. of the standard normal distribution in your solution.

4. Let $\mu : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}_+$. The goal of this exercise is to find the *stationary distribution* of the Markov process which arises as the solution of the SDE

   $$\mathrm{d}X_t = \mu(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t,$$

   i.e., to find the probability density function (p.d.f.) $\pi : \mathbb{R} \to \mathbb{R}_+$ such that if $\mathbb{P}(X_0 \in [x, x+\mathrm{d}x]) = \pi(x)\mathrm{d}x$ then $\mathbb{P}(X_t \in [x, x + \mathrm{d}x]) = \pi(x)\mathrm{d}x$ for all $t \geq 0$.

   (a) Use stochastic calculus to argue that if $\pi(\cdot)$ satisfies the above required stationarity property then it must also satisfy

   $$\int_{-\infty}^{\infty} \left(f'(x)\mu(x) + \frac{1}{2} f''(x)\sigma^2(x) \right) \cdot \pi(x)\, \mathrm{d}x = 0$$

   for any smooth and compactly supported test function $f : \mathbb{R} \to \mathbb{R}$.

   (b) Find the p.d.f. $\pi(\cdot)$ that satisfies the integral equation given in (a) for all test functions $f : \mathbb{R} \to \mathbb{R}$.