3.2 $\Omega = \{1,2,\ldots,6\}^3$ $(x_1, x_2, x_3) \in \Omega$

$\mathbb{P}$ IS UNIFORM ON Ω

$A := \{x_1 \neq x_2, x_2 \neq x_3, x_1 \neq x_3\}$ $A \subseteq \Omega$

THE CONDITIONAL DISTRIBUTION $\mathbb{P}(\cdot \mid A)$ IS UNIFORM ON A.

THUS $|A| = 6 \cdot 5 \cdot 4$

LET $B := \{x_1 = 6\} \cup \{x_2 = 6\} \cup \{x_3 = 6\}$

$$\mathbb{P}(B \mid A) = 1 - \mathbb{P}(B^c \mid A) = 1 - \frac{\mathbb{P}(A \cap B^c)}{\mathbb{P}(A)} =$$

$$= 1 - \frac{(5 \cdot 4 \cdot 3)/6^3}{(6 \cdot 5 \cdot 4)/6^3} = 1 - \frac{5 \cdot 4 \cdot 3}{6 \cdot 5 \cdot 4}$$

3.3 $A := \{$ WE USE DIE $\alpha\}$

$B := \{$ WE USE DIE $\beta\}$

$A \cup B = \Omega$, $A \cap B = \emptyset$, $\mathbb{P}(A) = \mathbb{P}(B) = \frac{1}{2}$

$C_k := \{k$'TH DIE ROLL GIVES RED$\}$

THE EVENTS $C_1, C_2, \ldots, C_n$ ARE NOT INDEPENDENT BUT THEY ARE CONDITIONALLY INDEPENDENT ONCE WE KNOW THE RESULT OF THE COIN TOSS.

a) $\mathbb{P}(C_n) = \underbrace{\mathbb{P}(C_n|A)}_{2/3} \cdot \frac{1}{2} + \underbrace{\mathbb{P}(C_n|B)}_{1/3} \cdot \frac{1}{2} = \frac{1}{2}$

(LAW OF TOTAL PROBABILITY)

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b) IS A SPECIAL CASE OF c)

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c) $$\mathbb{P}\left(C_n \middle| \bigcap_{i=1}^{n-1} C_i\right) = \frac{\mathbb{P}(C_1 \cap \ldots \cap C_n)}{\mathbb{P}(C_1 \cap \ldots \cap C_{n-1})} =$$

$$= \frac{\mathbb{P}(C_1 \cap \ldots \cap C_n|A) \cdot \frac{1}{2} + \mathbb{P}(C_1 \cap \ldots \cap C_n|B) \cdot \frac{1}{2}}{\mathbb{P}(C_1 \cap \ldots \cap C_{n-1}|A) \cdot \frac{1}{2} + \mathbb{P}(C_1 \cap \ldots \cap C_{n-1}|B) \cdot \frac{1}{2}}$$

CONDITIONAL INDEPENDENCE

$$= \frac{\left(\frac{2}{3}\right)^n \cdot \frac{1}{2} + \left(\frac{1}{3}\right)^n \cdot \frac{1}{2}}{\left(\frac{2}{3}\right)^{n-1} \cdot \frac{1}{2} + \left(\frac{1}{3}\right)^{n-1} \cdot \frac{1}{2}} = P_n$$

NOTE $\lim_{n \to \infty} P_n = \frac{2}{3}$ BECAUSE

$$P_n \approx \frac{\left(\frac{2}{3}\right)^n \cdot \frac{1}{2}}{\left(\frac{2}{3}\right)^{n-1} \cdot \frac{1}{2}} = \frac{2}{3}$$

BECAUSE IF n IS BIG AND WE SEE $n-1$ REDS THEN MOST LIKELY WE USE THE d DIE AND SO THE PROBAB. OF RED IS $\frac{2}{3}$

3.3 b) $\mathbb{P}(C_2 | C_1) = \frac{\left(\frac{2}{3}\right)^2 \cdot \frac{1}{2} + \left(\frac{1}{3}\right)^2 \cdot \frac{1}{2}}{1/2} = \frac{5}{9}$

THUS $\mathbb{P}(C_2 | C_1) \neq \mathbb{P}(C_2) = \frac{1}{2}$ THUS C_1 AND C_2 ARE NOT INDEPENDENT (BUT THEY ARE CONDITIONALLY INDEP. GIVEN A, AND THEY ARE COND. INDEP GIVEN B)

3.6 $A := \{$ ARMAND BAKED THE CAKE $\}$

$B := \{$ BENOIT — " — — " — $\}$

$C := \{$ CEDRIC — " — — " — $\}$

$R := \{$ THE CAKE IS RUINED $\}$

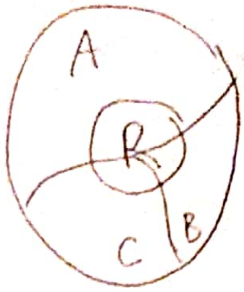
X = THE PERCENTAGE OF RUINED CAKES BAKED BY ARMAND

$X = 100 \cdot \mathbb{P}(A|R)$ $\quad \mathbb{P}(A|R) = \frac{\mathbb{P}(A \wedge R)}{\mathbb{P}(R)} = ☆$

WE KNOW: $\mathbb{P}(R|A) = 0.02 \quad \mathbb{P}(A) = 0.5$

$\mathbb{P}(R|B) = 0.03 \quad \mathbb{P}(B) = 0.3$

$\mathbb{P}(R|C) = 0.05 \quad \mathbb{P}(C) = 0.2$



A, B, C ARE A SET OF MUTUALLY EXCLUSIVE AND COLLECTIVELY EXHAUSTIVE EVENTS (I.E. $A \cap B = B \cap C = A \cap C = \emptyset$ AND $A \cup B \cup C = \Omega$)

THUS TOTAL LAW OF PROBABILITY

$$\mathbb{P}(R) = \underbrace{\mathbb{P}(R|A)}_{0.02} \cdot \underbrace{\mathbb{P}(A)}_{0.5} + \underbrace{\mathbb{P}(R|B)}_{0.03} \cdot \underbrace{\mathbb{P}(B)}_{0.3} + \underbrace{\mathbb{P}(R|C)}_{0.05} \cdot \underbrace{\mathbb{P}(C)}_{0.2}$$

$\mathbb{P}(R) = 0.029$, THUS

$$\text{☆} = \frac{\mathbb{P}(A \cap R)}{\mathbb{P}(R)} = \frac{\mathbb{P}(R|A) \cdot \mathbb{P}(A)}{\mathbb{P}(R)} = \frac{0.02 \cdot 0.5}{0.029} = \frac{10}{29}$$

THIS EXERCISE WAS AN EXAMPLE OF BAYES' THEOREM.

3.7 $D = \{\text{FETUS HAS DOWN SYNDROME}\}$

$T = \{\text{TEST SAYS} \quad -\text{''}- \quad -\text{''}- \}$

$\mathbb{P}(D) = \frac{1}{1000}$ $\quad \mathbb{P}(T^c | D) = \frac{1}{100}$

$\mathbb{P}(T | D^c) = \frac{5}{100}$

$\mathbb{P}(D|T) = ?$

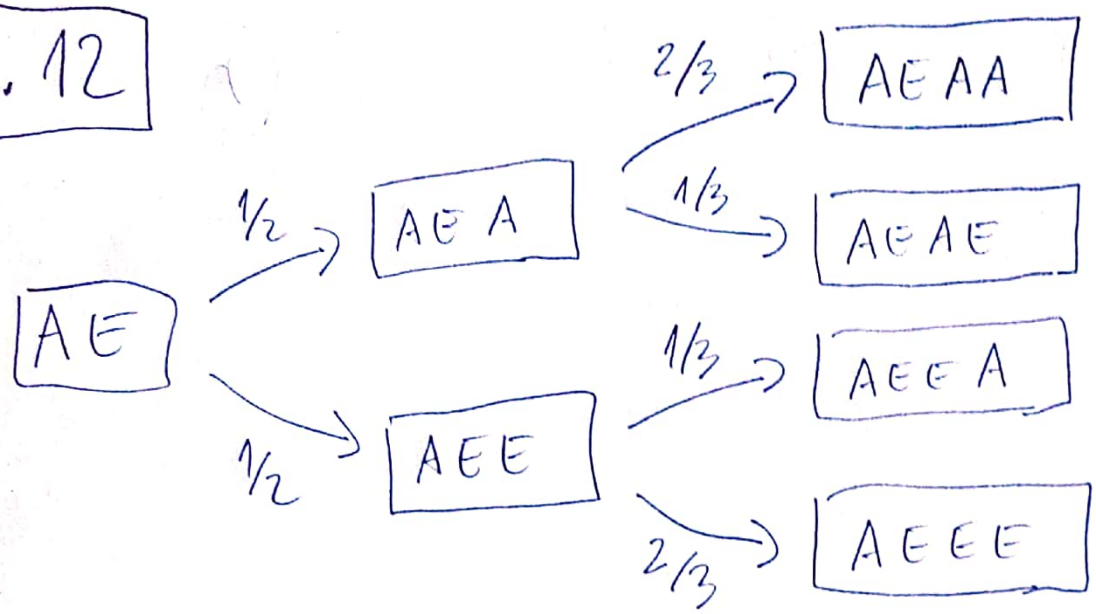
$$\mathbb{P}(D|T) = \frac{\mathbb{P}(D \cap T)}{\mathbb{P}(T)} \stackrel{\text{LAW OF TOTAL PROBAB.}}{=}$$

$$= \frac{\mathbb{P}(T|D) \cdot \mathbb{P}(D)}{\underbrace{\mathbb{P}(T|D)}_{\frac{99}{100}} \cdot \underbrace{\mathbb{P}(D)}_{\frac{1}{1000}} + \underbrace{\mathbb{P}(T|D^c)}_{\frac{5}{100}} \cdot \underbrace{\mathbb{P}(D^c)}_{\frac{999}{1000}}} = 0.01943\ldots$$

SO THE YOUNG COUPLE DOESN'T HAVE TO BE WORRIED TOO MUCH ABOUT THE POSITIVE TEST.

NAIVELY: OUT OF 1000 FETUSES, ONE HAS DOWN SYNDROME AND AROUND 50 IS A FALSE POSITIVE, SO THE FRACTION OF TRUE POSITIVES AMONG THE POSITIVES IS $\approx \frac{1}{51} \approx 0.02$

3.12



PÓLYA URN MODEL

$A_k^n :=$ {ADAM'S CLIQUE CONSISTS OF k PEOPLE WHEN THE FACEBOOK GROUP HAS n MEMBERS}

a) $\mathbb{P}(A_1^4) = \mathbb{P}(AEEE) = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$

$\mathbb{P}(A_2^4) = \mathbb{P}(AEAE) + \mathbb{P}(AEEA) = \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{3}$

$\mathbb{P}(A_3^4) = \mathbb{P}(AEAA) = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$

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b) CONJECTURE: $\mathbb{P}(A_k^n) = \frac{1}{n-1}$, $1 \le k \le n-1$

HOLDS FOR $n=2$, $n=3$ AND $n=4$ (BY a))

$$\mathbb{P}(A_k^{n+1}) = \underbrace{\mathbb{P}(A_k^{n+1} | A_k^n)}_{1-\frac{k}{n}} \cdot \underbrace{\mathbb{P}(A_k^n)}_{\frac{1}{n-1}} + \underbrace{\mathbb{P}(A_k^{n+1} | A_{k-1}^n)}_{\frac{k-1}{n}} \cdot \underbrace{\mathbb{P}(A_{k-1}^n)}_{\frac{1}{n-1}}$$

$$= \frac{n-1}{n} \cdot \frac{1}{n-1} = \frac{1}{n} \checkmark$$

3.10 YOU WIN WITH THE „SWITCH" STRATEGY IF AND ONLY IF YOU LOSE WITH THE „STAY" STRATEGY.

P(LOSE WITH „STAY" STRATEGY) $= \frac{2}{3}$

THUS P(WIN WITH „SWITCH") $= \frac{2}{3}$

THUS IT IS A GOOD IDEA TO SWITCH.

3.13 $A = \{$ PASSANGER 100 SITS IN HER OWN SEAT $\}$

P(A) DOES NOT CHANGE IF PASSENGERS 2, 3, ..., 99 EVICT PASSANGER 1 AGAIN AND AGAIN IF THEY FIND HIM SITTING IN THEIR SEATS, AND IT IS PASS. 1 WHO PICKS A NEW EMPTY SEAT AGAIN AND AGAIN.

NOW PASSANGERS 2, 3, ..., 99 ALL SIT IN THEIR OWN SEATS WHEN PASS. 100 ARRIVES.

EASY TO SEE: AFTER THE ARRIVAL OF PASS. i, $i = 1, 2, 3, \ldots, 99$, PASS. 1 IS UNIFORMLY DISTRIBUTED ON THE REMAINING EMPTY SEATS. THUS WHEN PASS. 100 ARRIVES, PASS 1. EITHER SITS AT HIS SEAT OR HER SEAT, WITH $\frac{1}{2}$-$\frac{1}{2}$ CHANCE. THUS P(A) $= \frac{1}{2}$