Probability 1 – Exercises

Tutorial no. 4 — 26th September 2024

4.1 A couple are in a team on a gameshow, and are asked a question. Both of them have p probability independently to know the answer. Which strategy is better?

(a) One of them are chosen randomly, and they answer without talking to the other.

(b) Both of them think of an answer, and if they agree they give that answer, otherwise they decide using a coin.

4.2 We toss a coin 3 times. Let A be the event that there is both at least one heads and at least one tails tossed. Let B be the event, that there is at most one tails in the outcomes. Are the two events independent?

4.3 We deal all the 52 cards of a French deck one-by-one. Which is the most likely position to see the second Ace?

HW 4.4 (2 points) We are throwing a standard die until the second 6. How many times do we have to throw the die most likely?

♣ **4.5** (3 points) Let's pick a random number from the set $\{1, 2, 3, \ldots, n\}$ with an equal probability. Let A_p be the event that the chosen number is divisible by p, which is a prime.

(a) Let's show that if $p_1, p_2, \ldots, p_k$ are prime numbers and n is divisible by $p_1, p_2, \ldots, p_k$, then $A_{p_1}, A_{p_2}, \ldots, A_{p_k}$ events are (totally) independent.

(b) Let C_n be the event that the randomly chosen number is a relative prime to n. Prove that

$$\mathbb{P}(C_n) = \prod_{p \text{ prime}, p|n} \left(1 - \frac{1}{p}\right)$$

4.6 We put a knight on a random position on a chess board. What is the expected value of the possible moves from that position?

4.7 In a casino, we play the following game: we bet a number from one to six, then a die is thrown three times. If our number appeared at least once then we get a peso for each appearance of our number, but we have to pay one peso if our number did not appear at all. Is the game fair?

HW 4.8 (2 points) Andrew and Bob are playing a game. There are 6 red and 6 green balls in an urn. They draw 2 balls from the urn. If they're both the same color, Andrew pays Bob 1000 forints. If they're different colors, Bob pays Andrew x forint. How much should x be to make the game fair?

4.9 (St. Petersburg paradox) We toss a coin until it lands on heads. If the n^{th} toss is heads, the player wins 2^n forint. Show that the expected value of the reward is infinite.

(a) Is it worth paying 1 million Ft to play the game?

(b) Is it worth paying 1 million Ft per game if we can play as much as we want, and only have to pay after we decide to stop?

4.10 A student has to fill out a test with 20 questions that can be answered with either "yes" or "no". Let's assume that the student knows the answer to each question independently with probability p. With probability q they think they know the answer, but they don't, and with probability r they know that they don't know the answer ($p + q + r = 1$). If the student knows that they don't know the answer, they randomly write "yes" or "no" with 50-50% probability. What is the probability that they answer at least 19 questions correctly?

4.11 Among five players, A, B, C, D, E, the numbers from 1 to 5 are distributed randomly, without repetition. First, A and B compete: whoever has the higher number moves on to play against the next opponent. The one who advances in this way will now face C, then the one who advances from among them will compete with D, then the winner battles with E. Let X be the number of rounds that A wins. Determine the distribution and expected value of X.

HW 4.12 (3 points) A class of 8 women and 5 men wrote a test. Let's assume that there are no ties in their test scores, and order them from best to worst, with each of the orderings having equal probability. Let X denote the placing of the highest ranking woman (eg. $X = 1$ means that the best test was written by a woman). Determine the distribution and the expected value of X.

4.13 Show that for a non-negative, integer valued random variable N, where $\mathbb{E}(N) < \infty$

$$\mathbb{E}(N) = \sum_{i=1}^{\infty} P(N \geq i)$$

4.14 There are 5 players; A,B,C,D,E and we give each of them a number between 1 and 5, without repetition (they can't have the same number). First, A and B fight, and whoever has the higher number goes onto the next round. Then, the person who won the previous round fights with C, and the person with the higher number wins. Similarly with D and E. Let X be the number of rounds A wins. What is the distribution and expected value of X?

4.15 On 4 buses 148 students are travelling all together. The number of students on each bus is 40, 33, 25 and 50, respectively. Choose a student uniformly at random, and let X denote the number of students on their bus. Also choose a bus driver uniformly at random, and let Y denote the number of students on their bus.

(a) Which is larger, $\mathbb{E}(X)$ or $\mathbb{E}(Y)$?

(b) Calculate $\mathbb{E}(X)$ and $\mathbb{E}(Y)$.

(c) Calculate $\mathbb{D}^2(X)$ and $\mathbb{D}^2(Y)$.

4.16 Anna and Bianca have two regular dice, one white and one yellow, and they play according to the following rules. First Anna rolls the two dice, and if the two dice have the same number on them, she wins. If Anna rolls different numbers then Bianca gets the dice and now she rolls: if she manages to roll at least one six, she wins. If not then there is no winner in the first round and they repeat the same process, so they roll alternately until one of them wins.

(a) Denote by X the total number of rounds that they play. What is the distribution and the expected value of X?

(b) What is the probability that the entire game ends with Anna winning?

HW 4.17 (3 points) We have two dice, and we're only focusing on the sum. We keep throwing until the sum of the two dice is either 7 or 10.

(a) What is the probability that we stop on 7?

(b) Prove that the number of throws and the sum when we stop are independent. In other words, show that if

$$A_k = \{\text{We throw k times}\} \qquad \text{and} \qquad B = \{\text{We stop on 7}\}$$

then for every $k \geq 1$, A_k and B are independent events.

4.18 There are $m + 1$ urns with m balls in each. In the i-th urn, there are $i - 1$ red and $m - i + 1$ blue balls ($i = 1, \ldots, m + 1$). First, we choose an urn uniformly randomly, then we draw n balls from the chosen urn with replacement.

(a) What is the probability that we only draw red balls?

(b) Assuming that the first n draws were red, what is the probability that the $n + 1$-th draw will be red as well?

(c) Calculate the limit of the probability obtained in part (b) for fixed n as $m \to \infty$.