$\boxed{4.3}$ $X :=$ POSITION OF SECOND ACE

$p_k := \mathbb{P}(X = k)$, $k = 2, 3, \ldots, 50$

$$\boxed{p_k = \frac{(k-1)\cdot\binom{52-k}{2}}{\binom{52}{4}}}$$

POSITION OF 2ND ACE IS FIXED
$k-1$ OPTIONS FOR 1ST
$\binom{52-k}{2}$ OPTIONS FOR 3RD 4TH

MOST LIKELY POSITION: THE VALUE OF k FOR WHICH p_k IS MAXIMAL.

GUESS: SOMEWHERE NEAR $2\cdot\frac{52}{5} \approx 21$

TRICK: $q_k := \frac{p_{k+1}}{p_k} = \frac{k\cdot\binom{51-k}{2}}{(k-1)\cdot\binom{52-k}{2}} =$

$$= \frac{k\cdot(51-k)\cdot(50-k)}{(k-1)\cdot(52-k)\cdot(51-k)} = \frac{k\cdot(50-k)}{(k-1)\cdot(52-k)} \overset{?}{>} 1$$

AFTER SOME SIMPLIFICATION: $\frac{52}{3} \overset{?}{>} k$

$17 \overset{?}{\geq} k$ THUS $q_k > 1$ IF $k = 2, 3, \ldots, 17$ AND $q_k < 1$ IF $k = 18, 19, \ldots, 49$ THUS p_{18} IS THE BIGGEST ONE. OUR GUESS WAS WRONG.

$\boxed{4.7}$ X = THE NUMBER OF DICE ON WHICH OUR NUMBER APPEARED

$X \sim BIN\left(3, \frac{1}{6}\right)$ (BINOMIAL DISTRIBUTION)

$p_k := \mathbb{P}(X = k), \quad k = 0, 1, 2, 3$

$$p_k = \binom{3}{k} \cdot \left(\frac{1}{6}\right)^k \cdot \left(\frac{5}{6}\right)^{3-k}$$

WE KNOW: $\mathbb{E}(X) = 3 \cdot \frac{1}{6} = \frac{1}{2}$

$$\mathbb{E}(X) = 0 \cdot p_0 + 1 \cdot p_1 + 2 \cdot p_2 + 3 \cdot p_3$$

MY NET PROFIT: Z_1 PESOS

$$Z_1 = \begin{cases} X & \text{IF } X \geq 1 \\ -1 & \text{IF } X = 0 \end{cases}$$

$$\mathbb{E}(Z_1) = -1 \cdot p_0 + 1 \cdot p_1 + 2 \cdot p_2 + 3 \cdot p_3$$

$$= \mathbb{E}(X) - p_0 = \frac{1}{2} - \left(\frac{5}{6}\right)^3 = \frac{-17}{216}$$

SO THIS GAME IS UNFAIR TO ME BECAUSE MY EXPECTED NET PROFIT IS NEGATIVE.

$\boxed{4.11}$ POSSIBLE VALUES OF X : $0, 1, 2, 3, 4$

WANT : $\mathbb{P}(X = k)$, $k = 0, 1, 2, 3, 4$

IT IS EASIER TO FIRST CALCULATE

$\mathbb{P}(X \geq k)$ FOR $k = 0, 1, 2, 3, 4$

$\mathbb{P}(X \geq k) = \mathbb{P}($ A WINS FIRST k ROUNDS $)$

$= \mathbb{P}($ A IS BETTER THAN THE OTHER k PLAYERS $)$

$= \frac{1}{k+1}$ BECAUSE EVERY RELATIVE ORDER OF THESE $k+1$ PLAYERS IS EQUALLY LIKELY.

<u>NOTE</u>: IF $\boxed{A \subseteq B}$ THEN $\boxed{\mathbb{P}(B \setminus A) = \mathbb{P}(B) - \mathbb{P}(A)}$

$$p_k := \mathbb{P}(X = k) = \mathbb{P}(X \geq k) - \mathbb{P}(X \geq k+1)$$

$$= \frac{1}{k+1} - \frac{1}{k+2} = \frac{1}{(k+1)\cdot(k+2)}$$

IF $k = 0, 1, 2, 3$

BUT IF $k = 4$: $\mathbb{P}(X = 4) = \mathbb{P}(X \geq 4) = \frac{1}{5}$

T.B.C.

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$$E(X) = \sum_{k=0}^{4} k \cdot p_k = 0 \cdot \frac{1}{1 \cdot 2} + 1 \cdot \frac{1}{2 \cdot 3} + 2 \cdot \frac{1}{3 \cdot 4} + 3 \cdot \frac{1}{4 \cdot 5} + 4 \cdot \frac{1}{5} = \frac{77}{60}$$

NOTE:

$$\sum_{k=1}^{4} P(X \geq k) = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{77}{60}$$

p_1			
p_2	p_2		
p_3	p_3	p_3	
p_4	p_4	p_4	p_4

IF YOU ADD THEM ROW-WISE

$$\sum_{k=0}^{4} k \cdot p_k$$

IF YOU ADD THEM COLUMN-WISE

$\boxed{4.16}$ FIRST ROUND:

$A := \{\text{ANN WINS}\}$ $B := \{\text{BIANCA WINS}\}$

$C := \{\text{NEITHER OF THEM WINS IN FIRST ROUND}\}$

A, B, C FORM A PARTITION OF Ω

$$\mathbb{P}(A) = \frac{1}{6} \quad \mathbb{P}(B) = \frac{5}{6} \cdot \underbrace{\mathbb{P}(\text{AT LEAST ONE } 6)}_{= 1 - \left(\frac{5}{6}\right)^2}$$

$$\mathbb{P}(B) = \frac{55}{216} \quad \mathbb{P}(C) = \frac{5}{6} \cdot \left(\frac{5}{6}\right)^2 = \frac{125}{216}$$

a) X HAS GEOMETRIC DISTRIBUTION

$X \sim GEO(p)$ WHERE $p = 1 - \mathbb{P}(C) = \frac{91}{216}$

KNOWN FACT: $\mathbb{E}(X) = \frac{1}{p} = \frac{216}{91}$

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b) $$\mathbb{P}(\text{ANN WINS}) = \sum_{k=1}^{\infty} \mathbb{P}\left(\begin{array}{c}\text{ANN WINS IN} \\ \text{THE } k\text{'TH ROUND}\end{array}\right) =$$

$$= \sum_{k=1}^{\infty} \mathbb{P}(C)^{k-1} \cdot \mathbb{P}(A) = \mathbb{P}(A) \cdot \frac{1}{1 - \mathbb{P}(C)} = \frac{36}{91}$$

<u>NOTE</u>: $$\mathbb{P}(\text{ANN WINS}) = \frac{\mathbb{P}(A)}{\mathbb{P}(A) + \mathbb{P}(B)} = \frac{\mathbb{P}(A)}{\mathbb{P}(A \cup B)} =$$

$= \mathbb{P}(A \mid A \cup B) =$ THE CONDITIONAL PROBABILITY THAT ANN WINS GIVEN THAT SOMEONE WINS IN THE 1ST ROUND

$\boxed{4.2}$ $\mathbb{P}(A) = \frac{8-2}{8} = \frac{3}{4}$

$\mathbb{P}(B) = \frac{1+3}{8} = \frac{1}{2}$ $\mathbb{P}(A \cap B) = \frac{3}{8}$

$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$, THUS A AND B ARE INDEPENDENT.

HIDDEN MEANING OF THIS EXERCISE:

Ω = THREE COINS

$\Psi : \Omega \to \Omega$: FLIP EACH COIN.

$\boxed{\Psi = \Psi^{-1}}$ $\boxed{\mathbb{P}(C) = \mathbb{P}(\Psi(C))}$ FOR EACH C

$\boxed{A = \Psi(A)}$: SYMMETRIC EVENT

$\boxed{B^c = \Psi(B)}$: ANTI-SYMMETRIC EVENT

THEOREM: ANY SYMMETRIC EVENT IS INDEPENDENT OF ANY ANTI-SYMMETRIC EVENT.

PROOF: $\mathbb{P}(B) = \mathbb{P}(\Psi(B)) = \mathbb{P}(B^c)$ THUS $\mathbb{P}(B) = \frac{1}{2}$

$\mathbb{P}(A \cap B) = \mathbb{P}(\Psi(A \cap B)) = \mathbb{P}(\Psi(A) \cap \Psi(B)) = \mathbb{P}(A \cap B^c)$

BUT $\mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^c) = \mathbb{P}(A)$ THUS

$\mathbb{P}(A \cap B) = \frac{1}{2}\mathbb{P}(A) = \mathbb{P}(B) \cdot \mathbb{P}(A)$: INDEPENDENCE!

4.10

$$p^* := \mathbb{P}(\text{STUDENT ANSWERS THE 1ST QUESTION CORRECTLY}) =$$

LAW OF TOTAL PROBABILITY

$$= p \cdot 1 + q \cdot 0 + r \cdot \frac{1}{2}$$

$$= p + \frac{r}{2}$$

X = NUMBER OF CORRECT ANSWERS

$X \sim \mathrm{BIN}(20, p^*)$, THUS

$$\mathbb{P}(X = r) = \binom{20}{r} \cdot (p^*)^r \cdot (1-p^*)^{20-r}, \quad r = 0, 1, \ldots, 20$$

$$\mathbb{P}(X \geq 19) = \underbrace{\mathbb{P}(X = 19)}_{20 \cdot (p^*)^{19} \cdot (1-p^*)} + \underbrace{\mathbb{P}(X = 20)}_{(p^*)^{20}}$$