<u>DEF</u>: X HAS POISSON DISTRIBUTION WITH MEAN λ (OR BRIEFLY: $X \sim POI(\lambda)$)

IF $\boxed{\mathbb{P}(X = k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!} \quad k = 0, 1, 2, 3, \ldots}$ $\boxed{\mathbb{E}(X) = \lambda}$

<u>FACT</u>: IF $X \sim BIN(n, p)$, WHERE n IS „BIG", BUT $n \cdot p$ IS „NOT BIG" THEN WE CAN „PRETEND" THAT $\boxed{X \sim POI(\lambda)}$

WHERE $\boxed{\lambda = n \cdot p}$

(POISSON APPROXIMATION OF BINOMIAL)

<u>WHY</u>? E.G. IF $X \sim BIN(200, 0.01)$

THEN

$$\mathbb{P}(X = 3) = \binom{200}{3} \cdot (0.01)^3 \cdot (1 - 0.01)^{197} =$$

$$\underbrace{\frac{200 \cdot 199 \cdot 198}{3!} \cdot \frac{1}{100} \cdot \frac{1}{100} \cdot \frac{1}{100}}_{\approx \frac{2^3}{3!}} \cdot \underbrace{\left(1 - \frac{1}{100}\right)^{197}}_{\approx e^{-2}}$$

$\boxed{5.1}$ X = NUMBER OF LEFT-HANDED PEOPLE IN OUR SAMPLE

$$X \sim BIN(\underbrace{200}_{n}, \underbrace{\tfrac{1}{100}}_{p})$$

$\boxed{n \text{ BIG}}$ $\boxed{n \cdot p = 2}$ NOT BIG

THUS POI APPROX: $X' \sim POI(2)$

$$\mathbb{P}(X \geq 4) = 1 - \mathbb{P}(X' \leq 3)$$
$$= 1 - e^{-2} - e^{-2} \cdot 2 - e^{-2} \cdot \frac{2^2}{2!} - e^{-2} \cdot \frac{2^3}{3!}$$
$$= 1 - \frac{19}{3} \cdot e^{-2} \approx 0.143$$

<u>FACT</u>: (THINNING OF POISSON)

IF I HAVE $POI(\lambda)$ PARTICLES AND EVERY PARTICLE IS KEPT WITH PROBABILITY p (INDEPENDENTLY OF EACH OTHER) THEN THE NUMBER OF KEPT PARTICLES HAS

$POI(p \cdot \lambda)$ DISTRIBUTION.

$\boxed{5.5}$ X = NUMBER OF OBSTACLES KNOCKED DOWN IN ONE LAP

$X \sim POI(\lambda)$ BY POI. APPROX OF BINOM.,
BUT WHAT IS λ?

$0.1 = \mathbb{P}(X = 0) = e^{-\lambda}$, THUS $\lambda = \ln(10) \approx 2.3$

(THE EXPECTED NUMBER OF OBSTACLES KNOCKED DOWN IN ONE LAP IS ≈ 2.3)

a) $\mathbb{P}(X \geq 4) = 1 - \mathbb{P}(X \leq 3) =$

$$= 1 - e^{-\lambda} - e^{-\lambda} \cdot \lambda - e^{-\lambda} \cdot \frac{\lambda^2}{2!} - e^{-\lambda} \cdot \frac{\lambda^3}{3!}$$

$$= 1 - \frac{1}{10} \cdot \left(1 + \ln(10) + \frac{\ln(10)^2}{2!} + \frac{\ln(10)^3}{3!}\right)$$

$$\approx 0.201$$

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b) Y = NUMBER OF INJURIES IN ONE LAP

THINNING: $Y \sim POI\left(\frac{1}{2} \cdot \ln(10)\right)$

$$\mathbb{P}(Y = 0) = e^{-\frac{1}{2} \cdot \ln(10)} = \frac{1}{\sqrt{10}} \approx 0.316$$

$\boxed{5.6}$ X = NUMBER OF TICKS IN ONE RUNNER

THEN $X \sim POI(\lambda)$ (SINCE MANY TICKS TRY TO ATTACK THE RUNNER INDEPENDENTLY OF EACH OTHER AND EACH TICK SUCCEEDS WITH SMALL PROBABILITY)

IF THERE ARE K RUNNERS ALTOGETHER THEN:

EXPECTED NUMBER OF RUNNERS WITH k TICKS IN THEM: $K \cdot \mathbb{P}(X = k)$.

"LAW OF LARGE NUMBERS":

$$\boxed{K \cdot \mathbb{P}(X = 1) \approx 300} \qquad \boxed{K \cdot \mathbb{P}(X = 2) \approx 75}$$

$$\boxed{K \cdot e^{-\lambda} \cdot \lambda = 300} \qquad \boxed{K \cdot e^{-\lambda} \cdot \frac{\lambda^2}{2} = 75}$$

TWO EQUATIONS, TWO UNKNOWNS.

DIVIDE THE TWO EQUATIONS:

$$\boxed{\frac{75}{300} = \frac{\lambda}{2}} \quad \text{THUS} \quad \boxed{\lambda = \frac{1}{2}} \quad \text{THUS}$$

$$K \cdot e^{-1/2} \cdot \frac{1}{2} = 300 \implies K = 600 \cdot e^{1/2} \approx 989$$

$\mathbb{E}(a \cdot X + b \cdot Y) = a \cdot \mathbb{E}(X) + b \cdot \mathbb{E}(Y)$, $a, b \in \mathbb{R}$

LINEARITY OF EXPECTATION

$\mathbb{D}^2(X) = \mathrm{Var}(X) := \mathbb{E}((X - m)^2) = \mathbb{E}(X^2) - m^2$

(WHERE $m = \mathbb{E}(X)$)

STEINER'S THM.

FACT: $\mathrm{Var}(a \cdot X + b) = a^2 \cdot \mathrm{Var}(X)$

$\boxed{5.3}$ $X \sim \mathrm{UNI}\{1, 2, \dots, 6\}$

$\mathbb{E}(X) = \sum_{k=1}^{6} \frac{1}{6} \cdot k = 3.5$ (CENTER OF MASS)

$\mathrm{Var}(X) = \sum_{k=1}^{6} \frac{1}{6} \cdot (k - 3.5)^2 = \frac{1}{3} \cdot (0.5)^2 + \frac{1}{3} \cdot (1.5)^2 + \frac{1}{3} \cdot (2.5)^2$

≈ 2.91

$\mathbb{E}((X - m)^2)$ (ROTATIONAL INERTIA)

$\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$

$= \sum_{k=1}^{6} \frac{1}{6} \cdot k^2 - (3.5)^2 \approx 2.91$

$\boxed{5.8}$

a) X_1 = NUMBER OF ROLLS UNTIL 1ST 5

X_2 = NUMBER OF ADDITIONAL ROLLS UNTIL 2ND 5

$Y = X_1 + X_2$ = NUMBER OF ROLLS UNTIL 2ND 5

$X_1 \sim GEO\left(\frac{1}{6}\right)$, $X_2 \sim GEO\left(\frac{1}{6}\right)$

$$\mathbb{E}(Y) \overset{\text{LIN.}}{=\!=} \mathbb{E}(X_1) + \mathbb{E}(X_2) = \frac{1}{1/6} + \frac{1}{1/6} = 12$$

b) $\mathbb{P}(Y > 12) = ?$

LET X := NUMBER OF FIVES IN FIRST TWELVE ROLLS

$$? = \mathbb{P}(X \leq 1) = \mathbb{P}(X = 0) + \mathbb{P}(X = 1) = \ddot{\smile}$$

$X \sim BIN\left(12, \frac{1}{6}\right)$

$$\ddot{\smile} = \left(\frac{5}{6}\right)^{12} + 12 \cdot \frac{1}{6} \cdot \left(\frac{5}{6}\right)^{11} \approx 0.381$$

$\boxed{5.9}$ $\boxed{20 \text{ BLUE}}$ $\boxed{50 \text{ NOT BLUE}}$

X = NUMBER OF BLUE BALLS OUT OF 7

$$\mathbb{P}(X = k) = \frac{\binom{20}{k} \cdot \binom{50}{7-k}}{\binom{70}{7}}, \quad k = 0, 1, \ldots, 7$$

(HYPERGEOMETRIC DISTRIBUTION)

$\mathbb{E}(X) = ?$

LET $X_i := \begin{cases} 1 & \text{IF } i\text{'TH SELECTED BALL IS BLUE} \\ 0 & \text{OTHERWISE} \end{cases}$

$$X = X_1 + \ldots + X_7$$

$$\mathbb{E}(X) \overset{\text{LIN.}}{=\!=} \mathbb{E}(X_1) + \ldots + \mathbb{E}(X_7) = 7 \cdot \mathbb{E}(X_1)$$

$$\mathbb{E}(X_1) = \mathbb{P}(\text{FIRST SELECTED BALL IS BLUE})$$
$$= \frac{20}{70} = \frac{2}{7}$$

THUS $\mathbb{E}(X) = 7 \cdot \frac{2}{7} = 2$

$\boxed{5.11}$ $\mathbb{E}(X^2) = \mathbb{E}(X)^2 + Var(X) = 1^2 + 5 = 6$

a) $\mathbb{E}((2+X)^2) = \mathbb{E}(4 + 4X + X^2) \overset{LIN.}{=}$

$4 + 4 \cdot \mathbb{E}(X) + \mathbb{E}(X^2) = 4 + 4 \cdot 1 + 6 = 16$

b) $(\mathbb{E}(2+X))^2 = (2 + \mathbb{E}(X))^2 = (2+1)^2 = 9$

c) $Var(4 + 3 \cdot X) = 3^2 \cdot Var(X) = 9 \cdot 5 = 45$

$\boxed{5.17}$ $\mathbb{P}(\text{RAINY} \mid 17 \text{ ACCIDENTS}) =$

$$= \frac{\mathbb{P}(\text{RAINY AND } 17 \text{ ACC.})}{\mathbb{P}(17 \text{ ACC.})} =$$

LAW OF TOTAL PROB. IN DENOMINATOR

$$= \frac{\mathbb{P}(\text{RAINY}) \cdot \mathbb{P}(17 \text{ ACC.} \mid \text{RAINY})}{\mathbb{P}(\text{RAIN}) \cdot \mathbb{P}(17 \text{ ACC.} \mid \text{RAIN}) + \mathbb{P}(\text{SUNNY}) \cdot \mathbb{P}(17 \mid \text{SUNNY})}$$

$$= \frac{(0.6) \cdot e^{-20} \cdot \frac{(20)^{17}}{17!}}{(0.6) \cdot e^{-20} \cdot \frac{(20)^{17}}{17!} + (0.4) \cdot e^{-10} \cdot \frac{(10)^{17}}{17!}} \approx 0.9$$