$\boxed{6.1}$ $$\mathbb{P}(Z_1 = k) = \sum_{\ell=0}^{k} \mathbb{P}(X = \ell, Y = k-\ell) =$$

$$= \sum_{\ell=0}^{k} \mathbb{P}(X = \ell) \cdot \mathbb{P}(Y = k-\ell) =$$

$$= \sum_{\ell=0}^{k} e^{-\lambda} \cdot \frac{\lambda^{\ell}}{\ell!} \cdot e^{-\mu} \cdot \frac{\mu^{k-\ell}}{(k-\ell)!} =$$

BINOMIAL THM.

$$\frac{e^{-(\lambda+\mu)}}{k!} \cdot \sum_{\ell=0}^{k} \underbrace{\frac{k!}{\ell! \cdot (k-\ell)!}}_{\binom{k}{\ell}} \cdot \lambda^{\ell} \cdot \mu^{k-\ell} = \frac{e^{-(\lambda+\mu)}}{k!} \cdot (\lambda+\mu)^{k}$$

$\boxed{\text{MERGING OF POISSON}}$

$\boxed{6.3}$ THE TIMES OF ARRIVALS OF SPEEDING CARS FORM A POISSON POINT PROCESS OF INTENSITY λ:

λ = EXPECTED NUMBER OF SPEEDING CARS PER MINUTE $\boxed{\lambda = ?}$

NUMBER X OF SPEEDING CARS IN 5-MIN WINDOW HAS POI$(5 \cdot \lambda)$ DISTRIBUTION

$$\mathbb{P}(X = 0) = \mathbb{P}(X \neq 0) = \frac{1}{2}$$

$\boxed{e^{-5\cdot\lambda} = \frac{1}{2}}$ $\boxed{5\lambda = \ln(2)}$ $\boxed{\lambda = \frac{1}{5}\cdot\ln(2)}$

a) Y = NUMBER OF SPEEDING CARS IN 20-MINUTE WINDOW

$$Y \sim POI(20\cdot\lambda) \sim POI(\ln(16))$$

(i) $$\mathbb{P}(Y=2) = \underbrace{e^{-\ln(16)}}_{1/16}\cdot\frac{\ln(16)^2}{2}$$

(ii) $$\mathbb{P}(Y\geq 2) = 1 - \mathbb{P}(Y=0) - \mathbb{P}(Y=1)$$

$$= 1 - \frac{1}{16} - \frac{1}{16}\cdot\ln(16)$$

b) Y_x = NUMBER OF SPEEDING CARS IN X-MINUTE WINDOW

$$Y_x \sim POI(x\cdot\lambda)$$

$$0.95 = \mathbb{P}(Y_x \geq 1) \qquad 0.05 = \mathbb{P}(Y_x = 0) = e^{-\lambda\cdot x}$$

$$e^{\lambda\cdot x} = 20 \qquad x = \frac{\ln(20)}{\lambda} = 5\cdot\frac{\ln(20)}{\ln(2)} = 21.6$$

$\boxed{6.4}$ SOLUTION OF HINT:

$X \sim POI(\lambda)$. X = NUMBER OF GOLD ORES

Y = NUMBER OF DISCOVERED GOLD ORES

IF $X = n$ THEN $Y \sim BIN(n, p)$

$$\mathbb{P}(Y = k) = \sum_{n=k}^{\infty} \mathbb{P}(X = n) \cdot \mathbb{P}(Y = k \mid X = n) =$$

$\boxed{\text{LAW OF TOTAL PROB}}$

$$= \sum_{n=k}^{\infty} e^{-\lambda} \cdot \frac{\lambda^n}{n!} \cdot \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k} =$$

$$= \frac{e^{-\lambda}}{k!} \cdot \sum_{n=k}^{\infty} \lambda^n \cdot \frac{1}{(n-k)!} p^k \cdot (1-p)^{n-k}$$

$$= \frac{e^{-\lambda}}{k!} \cdot (\lambda \cdot p)^k \cdot \underbrace{\sum_{n=k}^{\infty} \frac{(\lambda \cdot (1-p))^{n-k}}{(n-k)!}}_{\sum_{l=0}^{\infty} \frac{(\lambda \cdot (1-p))^l}{l!} = e^{\lambda \cdot (1-p)}}$$

$$= e^{-\lambda \cdot p} \cdot \frac{(\lambda \cdot p)^k}{k!}$$

$\boxed{\text{THINNING OF POISSON}}$

$\boxed{6.4}$ NOW $\boxed{\lambda = 10}$ AND $\boxed{p = \frac{1}{50}}$ THUS

$Y \sim POI\left(10 \cdot \frac{1}{50}\right) \sim POI\left(\frac{1}{5}\right)$

a) $\mathbb{P}(Y = 1) = e^{-1/5} \cdot \frac{1}{5}$ b) $\mathbb{P}(Y \geq 1) = 1 - e^{-1/5}$

c) $\mathbb{P}(Y \leq 1) = e^{-1/5} \cdot \frac{(1/5)^0}{0!} + e^{-1/5} \cdot \frac{(1/5)^1}{1!} = e^{-1/6} \cdot \frac{6}{5}$

$\boxed{6.6}$ (2-DIMENSIONAL POISSON POINT PROCESS)

INTENSITY : 0.2 WHIRLPOOLS / km^2

DISTANCE FROM ITHACA: d km

AREA OF DANGER ZONE : $(0.3) \cdot d$ km^2

EXPECTED NUMBER OF WHIRLPOOLS IN THE DANGER ZONE : $(0.2) \cdot (0.3) \cdot d$

X = NUMBER OF WHIRLPOOLS IN DANGER ZONE

$X \sim POI\left(\frac{6}{100} \cdot d\right)$ $0{,}01 = \mathbb{P}(X = 0) = e^{-\frac{6}{100} \cdot d}$

$\frac{6}{100} \cdot d = \ln(100)$ $d = \frac{100}{6} \cdot \ln(100) \approx 76.75$

$\boxed{6.7}$

a) ONE SQUARE OF CHOC. HAS POI(λ) NUTS

$$e^{-\lambda} = \frac{1}{2} \implies \boxed{\lambda = \ln(2)}$$

MERGING OF POISSONS: ONE BAR OF CHOC. HAS POI$(15 \cdot \ln(2))$ NUTS

b) ASSUME THAT NUTS AND RAISINS ARE INDEPENDENT.

MEAN NUMBER OF RAISINS IN 1 BAR: 30

— " — — " — — " — IN 1 SQUARE: 2

IN TWO SQUARES: POI(4) RAISINS AND POI$(2 \cdot \ln(2))$ NUTS, THUS BY MERGING OF POISSONS:

IF X' = COMBINED NUMBER OF NUTS & RAISINS IN 2 SQUARES

THEN $X' \sim \mathrm{POI}(4 + 2 \cdot \ln(2)) = \mathrm{POI}(\mu)$

$$\mathbb{P}(X' < 2) = \underbrace{\mathbb{P}(X' = 0)}_{e^{-\mu}} + \underbrace{\mathbb{P}(X' = 1)}_{e^{-\mu} \cdot \mu}$$

6.8 „COUPON COLLECTOR'S PROBLEM"

THERE ARE SIX TYPES OF COUPONS AND WE WANT ONE FROM EACH TYPE. IF WE THROW k THEN WE GET ONE k-TYPE COUPON $(k = 1, 2, \dots, 6)$

IF WE HAVE ALREDY COLLECTED m DIFFERENT TYPES OF COUPONS THEN LET Y_m DENOTE THE NUMBER OF ADDITIONAL THROWS WE NEED UNTIL WE SEE A NEW TYPE OF COUPON.

WANT: $\mathbb{E}(Y_0 + Y_1 + \dots + Y_5) = ?$

$Y_m \sim GEO\left(\frac{6-m}{6}\right)$ BECAUSE IF I HAVE ALREDY SEEN m TYPES THEN THERE ARE $6-m$ POTENTIAL NEW TYPES.

$$\mathbb{E}(Y_m) = \frac{6}{6-m}$$

$$? \underset{\text{LINEARITY OF EXPECTATION}}{=} \mathbb{E}(Y_0) + \dots + \mathbb{E}(Y_5) = \frac{6}{6} + \frac{6}{5} + \dots + \frac{6}{2} + \frac{6}{1}$$

$\boxed{6.10}$ $X :=$ NUMBER OF THROWS UNTIL 1ST ⚅

$Y :=$ — " — — " — — " — — " — 2ND ⚅

$\mathbb{P}(X = k \mid Y = m) = ?$ WHERE $0 < k < m$

$$\mathbb{P}(X = k \mid Y = m) = \frac{\mathbb{P}(X = k, Y = m)}{\mathbb{P}(Y = m)} = \frac{\left(\frac{5}{6}\right)^{k-1} \cdot \frac{1}{6} \cdot \left(\frac{5}{6}\right)^{m-k-1} \cdot \frac{1}{6}}{\sum_{\ell=1}^{m-1} \left(\frac{5}{6}\right)^{\ell-1} \cdot \frac{1}{6} \cdot \left(\frac{5}{6}\right)^{m-\ell-1} \cdot \frac{1}{6}} =$$

$$= \frac{\left(\frac{5}{6}\right)^{m-2} \cdot \left(\frac{1}{6}\right)^{2}}{(m-1) \cdot \left(\frac{5}{6}\right)^{m-2} \cdot \left(\frac{1}{6}\right)^{2}} = \frac{1}{m-1}$$ THUS

IF $Y = m$ THEN X IS UNIFORMLY DISTRIBUTED ON $\{1, 2, \dots, m-1\}$.

THUS, NO MATTER WHAT BERNADETTE GUESSES FROM THE NUMBERS $1, 2, \dots, m-1$, HER CHANCE OF WINNING IS $\frac{1}{m-1}$.

<u>NOTE</u>: Y HAS NEGATIVE BINOMIAL DISTRIBUTION (Y IS THE SUM OF TWO I.I.D. GEO(1/6) R.V.'S)