7.1 b) $F(x) = \exp\left(-A e^{-Bx}\right)$

If $A < 0$, then $F\left(-\frac{1}{B}\right) = \exp(-Ae) > e^0 = 1$ ↯

If $A = 0 \Rightarrow F(x) = 1$ constant ↯

$\rightarrow \underline{A > 0}$

If $B < 0 \Rightarrow F$ is monotone decreasing ↯

If $B = 0 \Rightarrow F(x) = e^{-A}$ constant ↯

What about $A, B > 0$?

$$\lim_{x \to \infty} F(x) = e^{-A e^{-B\infty}} = e^{-A \cdot 0} = 1 \checkmark$$

$$\lim_{x \to -\infty} F(x) = e^{-A e^{B\infty}} = e^{-A\infty} = 0 \checkmark$$

$F(x)$ is called Gumbel distribution

7.2 b, $f(x) = \begin{cases} c \cdot x^{-5}, & \text{if } x \geq 1. \\ 0 & \text{otherwise} \end{cases}$

If $c > 0$, then $f(x) \geq 0$

$$1 = \int_{-\infty}^{\infty} f(x)\,dx = c\int_{1}^{\infty} x^{-5}\,dx = c\left[-\tfrac{1}{4}x^{-4}\right]_1^{\infty} = \frac{c}{4} \Rightarrow \underline{\underline{c = 4}}$$

We need this

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x f(x)\,dx = 4\int_{1}^{\infty} x^{-4}\,dx = 4\left[-\tfrac{1}{3}x^{-3}\right]_1^{\infty} = \underline{\underline{\frac{4}{3}}}$$

$$\mathbb{P}\left(X \in \left[\tfrac{1}{2}, 2\right]\right) = \mathbb{P}(X \in [1, 2]) = 4\int_{1}^{2} x^{-4}\,dx = 4\left[-\tfrac{1}{4}x^{-4}\right]_1^{2} = \underline{\underline{1 - 2^{-4}}}$$

7.3 $f(x) = \begin{cases} 5(1-x)^4 & \text{if } x \in]0, 1[\\ 0 & \text{otherwise} \end{cases}$

$0{,}01 > \mathbb{P}(\text{running out of oil}) = \mathbb{P}(\text{sales} > \text{shipment}) = \mathbb{P}(X > \text{shipment})$

The station gets a shipment of $V \cdot 1000$ litres

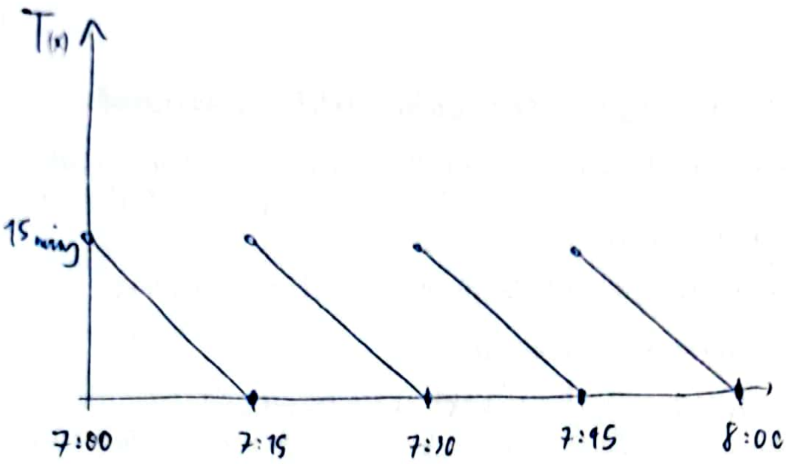
$$0{,}01 > \mathbb{P}(X > V) = 1 - \mathbb{P}(X \leq V) = 1 - \int_{-\infty}^{V} f(x)\,dx =$$

$$= 1 - 5\int_{0}^{V} (1-x)^4\,dx = 1 - 5\left[-\tfrac{1}{5}(1-x)^5\right]_0^{V} = 1 - (1 - (1-v)^5) =$$

$$= (1-v)^5 \quad \Rightarrow \quad \underline{\underline{V > 1 - (0{,}01)^{\frac{1}{5}} \approx 0{,}6}}$$

7.5

$T(x) := \{$The time we need to wait until the next bus comes$\}$



a) We arrive uniformly between 7:00 and 7:30

$\mathbb{P}(T(x) < 4) = ?$

We need to find subintervals where we wait less than 4 mins.

$I :=]7{:}11, 7{:}15] \cup]7{:}26, 7{:}30]$

The length of I : $|I| = 4+4 = 8$

$\Rightarrow \mathbb{P}(T(x) < 4) = \dfrac{8}{30} = \dfrac{4}{15}$

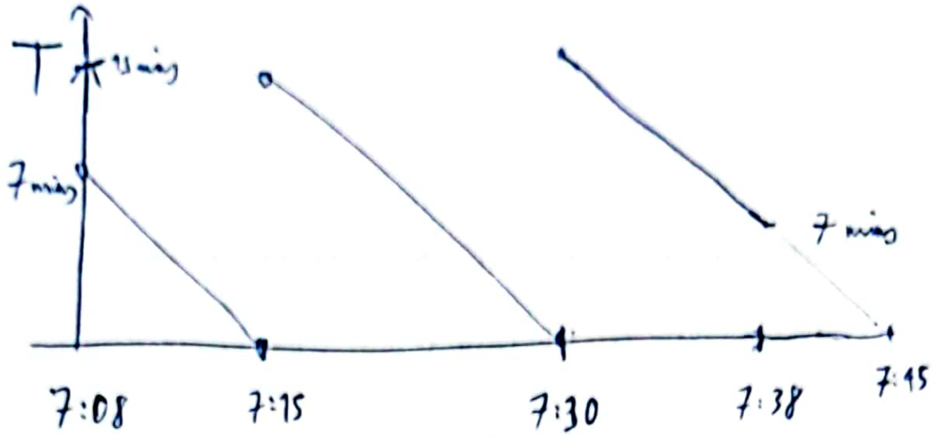
b) $\mathbb{P}(T(x) > 7) = ?$

$I := [7{:}00, 7{:}08[\cup [7{:}15, 7{:}23[$

$|I| = 8 + 8 = 16$

$\mathbb{P}(T(X) > 7) = \dfrac{16}{30} = \dfrac{8}{15}$

c/



We arrive uniformly between 7:08 and 7:38

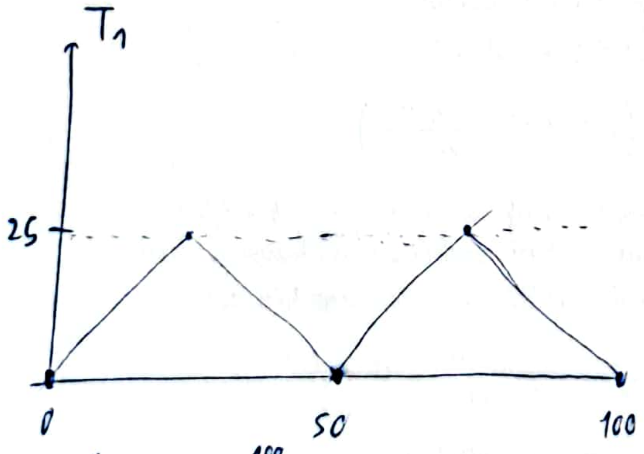
$$\mathbb{P}(T(x) < 4) = \frac{4+4}{30} = \frac{4}{15}$$

$$\mathbb{P}(T(x) > 7) = \frac{8+8}{30} = \frac{8}{15}$$

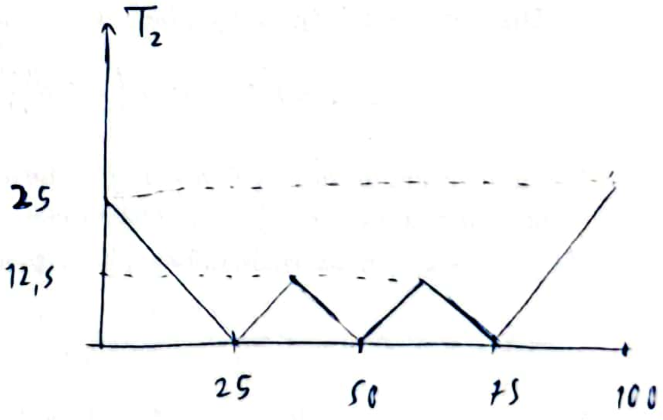
7.7 $X = \{\text{place where the bus breaks down}\} \sim Uni(0,100)$

$T(x) :=$ distance from the closest mechanic

1st case:



2nd case

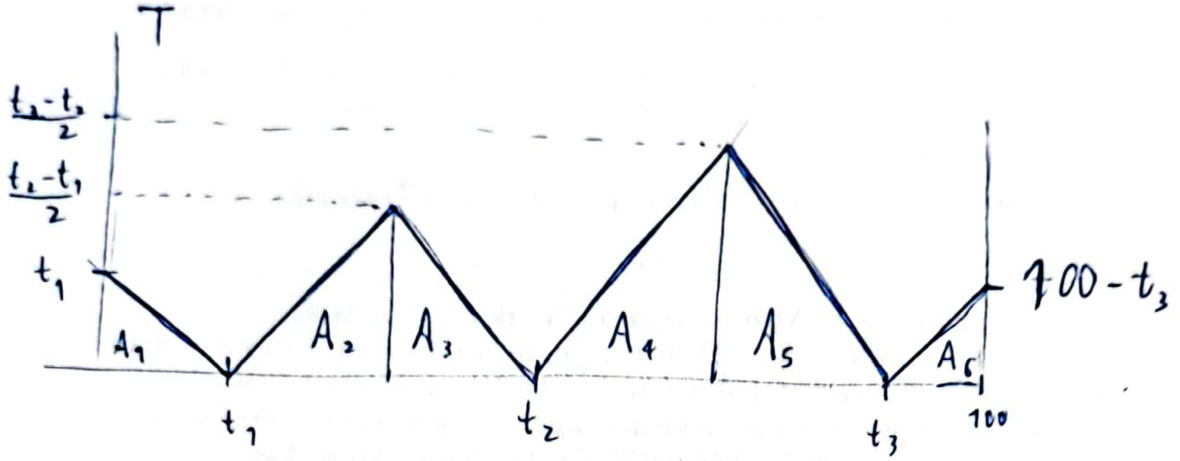


$$\mathbb{E}(T_1(X)) = \int_0^{100} T_1(x) \frac{1}{100}\, dx = \underline{12,5}$$

$$\mathbb{E}(T_2(X)) = \int_0^{100} T_2(x) \frac{1}{100}\, dx = \underline{9,375}$$

Best setup

If the mechanics are at $t_1, t_2, t_3 \in [0,100]$



A_i is the area under the ith branch of T

$$\mathbb{E}(T(X)) = \frac{1}{100}\int_0^{100} T(x)\,dx = \frac{1}{100}\sum_{i=1}^{6} A_i$$

We know from the arithmetic-geometric inequality, that $\mathbb{E}(T(X))$ is minimal, if $A_i = A_j \quad \forall i,j$

$$\Rightarrow \frac{t_1^2}{2} = A_1 = A_2 = \frac{\left(\frac{t_2-t_1}{2}\right)^2}{2} = A_4 = \frac{\left(\frac{t_3-t_2}{2}\right)^2}{2} = A_6 = \frac{(100-t_3)^2}{2}$$

$$\Rightarrow t_1 = \frac{t_2-t_1}{2} = \frac{t_3-t_2}{2} = 100-t_3$$

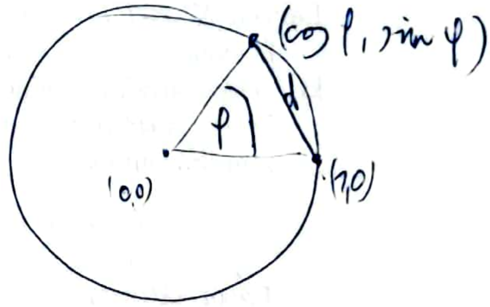
$$\Rightarrow t_1 = \frac{1}{6}100 \quad , t_2 = \frac{1}{2}100 \quad , t_3 = \frac{5}{6}\cdot 100$$

7.8 Due to rotational symmetry, we can assume that the first point is $(1,0)$

If the other point is $(\cos\varphi, \sin\varphi)$, then their distance is

$$d^2 = (\cos\varphi - 1)^2 + (\sin\varphi)^2 = \cos^2\varphi + 1 - 2\cos\varphi + \sin^2\varphi =$$

$$= 2(1-\cos\varphi)$$



As $\cos\varphi = \cos(-\varphi)$

$\varphi \sim Uni(0,\pi)$

We know, that $\mathbb{P}(d \in [0,2]) = 1$

For $d_0 \in [0,2]$

$$\mathbb{P}(d \le d_0) = \mathbb{P}(d^2 \le d_0^2) = \mathbb{P}(2(1-\cos\varphi) \le d_0^2) =$$

$$= \mathbb{P}\left(1 - \frac{d_0^2}{2} \le \cos\varphi\right) = \cancel{\mathbb{P}\left(\varphi \in \left]\operatorname{acos}\left(1-\frac{d_0^2}{2}\right), \pi\right[\right) =}$$

$$= \mathbb{P}\left(\varphi \le \operatorname{acos}\left(1-\frac{d_0^2}{2}\right)\right) = \frac{1}{\pi}\operatorname{acos}\left(1-\frac{d_0^2}{2}\right)$$

since acos is monotone decreasing

7.9 $X \sim \text{Uni}(B^3)$ $\quad B^3 = \{x \in \mathbb{R}^3 \mid |x| < 1\}$

$$f(x) = \frac{1}{\text{Vol}(B^3)} = \frac{3}{4\pi}$$

Volume of ball with radius r: $\frac{4r^3}{3}\pi$

ξ = (distance from the center)

$$F(r) = \mathbb{P}(\xi \leq r) = \frac{\frac{4}{3}r^3\pi}{\frac{4}{3}\pi} = r^3 \quad \forall r \in [0,1]$$

$$\mathbb{E}(\xi) = \int_0^\infty r f(r)\,dr = \int_0^{1} r\, DF(r)\,dr = \int_0^1 r \cdot 3r^2\,dr =$$

$$= 3\int_0^1 r^3\,dr = 3\left[\frac{1}{4}r^4\right]_0^1 = \underline{\underline{\frac{3}{4}}}$$