STANDARD NORMAL DISTRIBUTION: $X \sim N(0,1)$

$$P(X \le x) = \Phi(x), \quad \Phi(-x) = 1 - \Phi(x)$$

$$\Phi(x) = \int_{-\infty}^{x} \varphi(s)\,ds, \quad \varphi(x) = \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{x^2}{2}\right)$$

IF $Y = \mu + \sigma \cdot X$ THEN $Y \sim N(\mu, \sigma^2)$

$$E(Y) = \mu, \quad Var(Y) = \sigma^2$$

$$P(Y \le x) = P(\mu + \sigma \cdot X \le x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

$\boxed{8.1}$ $Y \sim N(100, 15^2)$, $X := \frac{1}{15} \cdot (Y - 100) \sim N(0,1)$

a) $P(95 \le Y \le 110) = P(-5 \le Y - 100 \le 10) =$

$P\left(-\frac{1}{3} \le X \le \frac{2}{3}\right) = P\left(X \le \frac{2}{3}\right) - P\left(X \le -\frac{1}{3}\right) =$

$= \Phi\left(\frac{2}{3}\right) - \Phi\left(-\frac{1}{3}\right) = \Phi\left(\frac{2}{3}\right) - \left(1 - \Phi\left(\frac{1}{3}\right)\right) =$

$= \Phi\left(\frac{2}{3}\right) + \Phi\left(\frac{1}{3}\right) - 1 = 0.7486 + 0.6293 - 1 = 0.378$

FROM STANDARD NORMAL TABLE

THUS APPROX. 38% OF PEOPLE HAVE AN IQ BETWEEN 95 AND 110

b) $\frac{1}{2} = \mathbb{P}(100 - X \leq Y \leq 100 + X) = \mathbb{P}(-X \leq Y - 100 \leq X) =$

$= \mathbb{P}\left(-\frac{X}{15} \leq \mathcal{X} \leq \frac{X}{15}\right) = \Phi\left(\frac{X}{15}\right) - \Phi\left(-\frac{X}{15}\right) =$

$= \Phi\left(\frac{X}{15}\right) - \left(1 - \Phi\left(\frac{X}{15}\right)\right) = 2 \cdot \Phi\left(\frac{X}{15}\right) - 1$

FROM STANDARD NORMAL TABLE

THUS $\Phi\left(\frac{X}{15}\right) = \frac{3}{4} = 0.75$, $\Phi^{-1}(0.75) \approx 0.68$

THUS $\frac{X}{15} = 0.68$, THUS $\boxed{X = 10.2}$

THUS THE INTERVAL $[100 - 10.2, 100 + 10.2]$ CONTAINS THE IQ OF 50% OF HUMANITY.

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c) $\mathbb{P}(Y \geq 125) = \mathbb{P}\left(\mathcal{X} \geq \frac{5}{3}\right) = 1 - \Phi\left(\frac{5}{3}\right) =$

$= 1 - \Phi(1.66) = 1 - 0.9515 = 0.0485$

THUS IN A TOWN OF 2500 PEOPLE HAS $2500 \cdot 0{,}0485 = 121$ GENIUSES WITH AN IQ OVER 125.

$\boxed{8.3}$ FRENCH DECK OF CARDS: $4 \times 13 = 52$

POKER HAND: 5 CARDS SUIT HEIGHT

NUMBER OF POKER HANDS: $\binom{52}{5}$

NUMBER OF FULL HOUSE – " – : $13 \cdot 12 \cdot \binom{4}{3} \cdot \binom{4}{2}$

$$p := \mathbb{P}(\text{FULL HOUSE}) = \frac{13 \cdot 12 \cdot 3 \cdot 6}{\binom{52}{5}} = 0.00144 = p$$

X := NUMBER OF FULL HOUSE HANDS OUT OF 10^5

$X \sim BIN(n, p)$, $n = 10^5$, $p = 0.00144$

$\mathbb{P}(128 < X < 158) = ?$ $\quad n \cdot p = 144 \leftarrow$ BIG

SO NOW WE DO NOT USE POISSON APPROX.,
BUT NORMAL APPROX (I.E., DE MOIVRE-LAPLACE)

$$\mathbb{E}(X) = n \cdot p = 144 \quad \underbrace{\sqrt{Var(X)}}_{\text{STANDARD DEVIATION}} = \sqrt{n \cdot p \cdot (1-p)} \approx 12$$

DE MOIVRE: IF $Y := \frac{X - n \cdot p}{\sqrt{n \cdot p \cdot (1-p)}}$ THEN $\boxed{Y \approx N(0,1)}$

$$? = \mathbb{P}\left(\frac{128-144}{12} < Y < \frac{158-144}{12}\right) \approx \Phi(1.16) - \Phi(-1.33)$$

$$= \Phi(1.16) + \Phi(1.33) - 1 = 0.8776 + 0.9082 - 1$$
$$= 0.7858$$

$\boxed{8.4}$ X_n = NUMBER OF SMOKERS OUT OF n

$X_n \sim BIN(n, p)$ $\boxed{p' := X_n / n}$

WANT: n SUCH THAT $0.93 \leq \mathbb{P}\left(\left|\frac{X_n}{n} - p\right| \leq 0.02\right) =$

$= \mathbb{P}(|X_n - n \cdot p| \leq n \cdot 0.02) =$

$\mathbb{P}\left(\left|\frac{X_n - n \cdot p}{\sqrt{n \cdot p \cdot (1-p)}}\right| \leq \frac{\sqrt{n} \cdot 0.02}{\sqrt{p \cdot (1-p)}}\right) \approx$

$\boxed{\text{DE MOIVRE LAPLACE}}$

$\approx \mathbb{P}(|Y| \leq y) =$

$\boxed{\text{WHERE } Y \sim N(0,1) \text{ AND } y := \frac{\sqrt{n} \cdot 0.02}{\sqrt{p \cdot (1-p)}}}$

$\mathbb{P}(-y \leq Y \leq y) = \Phi(y) - \Phi(-y) = 2 \cdot \Phi(y) - 1 \geq 0.93$

WANT

WANT: $\Phi(y) \geq 0.965$

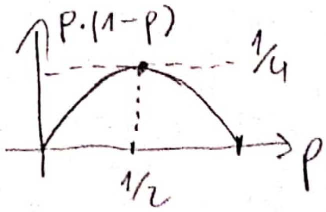
FROM NORMAL TABLE: $\Phi^{-1}(0.965) = 1.82$

THUS: WANT: $\boxed{y \geq 1.82}$ $\boxed{\frac{0.02 \cdot \sqrt{n}}{\sqrt{p \cdot (1-p)}} \geq 1.82}$ $\leftarrow$ WANT

WANT: $\boxed{n \geq 8281 \cdot p \cdot (1-p)}$ BUT WE DON'T KNOW THE VALUE OF p,

SO LET US CONSIDER THE WORST CASE:

$\max_{0 \leq p \leq 1} p \cdot (1-p) = \frac{1}{4}$ SINCE



T.B.C.

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THUS $m \geq 8281 \cdot \frac{1}{4} = 2070.25$

SO WE NEED TO ASK 2071 PEOPLE ABOUT THEIR SMOKING HABITS

$\boxed{8.6}$ a) X = ARRIVAL TIME OF NEXT METRO (IN MINUTES)

$X \sim EXP(\lambda)$, λ = INTENSITY OF PPP OF ARRIVAL TIMES

$\mathbb{E}(X) = \frac{1}{\lambda} = 2$ THUS $\boxed{\lambda = \frac{1}{2}}$

$\mathbb{P}(X \geq 5) = e^{-\lambda \cdot 5} = e^{-2.5} \approx 0.082$

b) BY THE <u>MEMORYLESS</u> PROPERTY OF EXPONENTIAL DISTRIBUTION, THE ANSWER WILL BE $e^{-2.5}$ AGAIN, BUT LET'S CALCULATE IT ANYWAY:

$$\mathbb{P}(X \geq 3+5 \mid X \geq 3) = \frac{\mathbb{P}(\{X \geq 8\} \cap \{X \geq 3\})}{\mathbb{P}(X \geq 3)} =$$

$$= \frac{\mathbb{P}(X \geq 8)}{\mathbb{P}(X \geq 3)} = \frac{e^{-\lambda \cdot 8}}{e^{-\lambda \cdot 3}} = e^{-\lambda \cdot 5} = e^{-2.5}$$ ✓

$\boxed{8.7}$ X = LIFETIME OF A PROBABILITIUM ATOM

$X \sim EXP(\lambda)$ $\mathbb{E}(X) = \frac{1}{\lambda} = 1 \Rightarrow \boxed{\lambda = 1}$

HALF-LIFE: X SUCH THAT $\mathbb{P}(X \geq x) = \frac{1}{2}$

$e^{-\lambda \cdot x} = \frac{1}{2}$ $e^{-x} = \frac{1}{2}$ $x = \ln(2)$

$\boxed{8.9}$ X = LIFETIME OF LIGHT BULB (IN YEARS)

$X \sim EXP(\lambda)$ $\boxed{\mathbb{P}(X \geq 1) = 0.9}$

$\boxed{\mathbb{P}(X \geq 1) = e^{-\lambda} = 0.9} \Rightarrow \boxed{\lambda = -\ln(0.9)}$

$x :=$ LENGTH OF WARRANTY PERIOD (YEARS)

WANT: $\boxed{\mathbb{P}(X \geq x) = 0.99}$

$\mathbb{P}(X \geq x) = e^{-\lambda \cdot x} = e^{\ln(0.9) \cdot x} = (0.9)^x = 0.99$

THUS $x = \frac{\ln(0.99)}{\ln(0.9)} = 0.095$ YEARS

THUS THE WARRANTY PERIOD IS ABOUT ONE MONTH LONG...