# Probability 1 – Exercises

## Tutorial no. 9 — 31th Oct 2024

**9.1** Assume that $Z$ is such a random variable that $2Z$ has the same distribution. What is the distribution of $Z$?

**9.2** Let $X$ be a continuous random variable, with CDF $F$. What is the distribution of $Y = F(X)$?

**9.3** Calculate the PDF of the following random variables:

(a) If $\xi$ is standard normal then what is the PDF of $X := \xi^2$?

**HW** (b) (1 point) If $\xi \sim \text{EXP}(\lambda)$, then what is the PDF of $X := 2\xi - 3$, and $Z := \sqrt[5]{\xi}$?

(c) If $\xi$ is uniform on the $[-1; 1]$ interval, then what is the PDF of $X := \xi^2$ and $U := cos(\pi\xi)$?

(d) If $\xi$ is exponential with parameter $\lambda$, then what is the PDF of $Y := e^{\xi}$?

**HW** (e) (2 points) If $\xi$ is uniform on $[-2; 3]$, then what is the PDF of $Z := \xi^2$?

**9.4** Let $X$ be a standard Cauchy random variable. (Recall that its PDF is $f(x) = \frac{1}{\pi(1+x^2)}$ for $x \in (-\infty, \infty)$.) Show that $X$ and $1/X$ have the same distribution.

♣ **9.5** (3 point) Let $X$ be a random variable with PDF $\frac{1}{\ln 2}\frac{1}{1+x}$ on the $[0, 1]$ interval, and 0 otherwise. What is the PDF of the fractional part of $\frac{1}{X}$?

**9.6** Let $\xi$ denote the distance of the point $X$ and the point of the plane with coordinates $(1; 1)$. Find the probabilty density function of $\xi$ if

(a) $X$ is uniformly distributed on the $[0, 1]$ interval of the x-axis.

(b) $X$ is uniformly distributed on the $[0, 2]$ interval of the x-axis.

**9.7** A stick of length $\ell$ is broken at two independently and uniformly distributed points. What is the expected value of the length of the shortest one out of the three pieces obtained in this way?

**9.8** We roll two independent fair dice. What is the joint probability weight function of $X$ and $Y$ if

**HW** (a) (2 points) $X$ is the maximum throw, $Y$ is their sum;

(b) $X$ is the first throw, $Y$ is the maximum of the two throws?

**9.9** From the Leaning Tower of Pisa, I drop a ball from height 40.5 meters. As we know from physics, in $t$ seconds it falls $gt^2/2$ meters, where $g = 9m/s^2$, hence it reaches the ground in exactly 3 seconds. Each of my two friends, Galileo and Isaac, takes a photo of the experiment, at independent random times with distribution Uni$[0; 3]$ sec.

(a) In expectation, how high is the ball on Isaac's photo?

(b) In expectation, how high is the ball on the photo that is taken later?

**HW 9.10** (3 points) Jack arrives to school with a delay of $X \sim \text{Uni}[0, 15]$ minutes. Jill arrives with a delay of $Y \sim \text{Uni}[0, 10]$ minutes, independently of Jack.

(a) What is the probability that Jill arrives earlier than Jack?

(b) What is the expectation of the difference $|X - Y|$?

**9.11** Let $(X, Y)$ be the pair of coordinates of uniformly chosen point of the unit disc of the plane. Determine the p.d.f. of the marginal distributions.

**9.12** Let $(X, Y)$ denote a pair of random variables with joint p.d.f. $f(x, y)$. Are $X$ and $Y$ independent?

(a) $f(x, y) = xe^{-(x+y)}\mathbb{1}[x > 0, y > 0]$.

(b) $f(x, y) = 2 \cdot \mathbb{1}[0 < x < y < 1]$.

**9.13** Let $X$ and $Y$ be random variables with joint probability density function

$$f(x,y) = \begin{cases} \frac{4}{5}(x+xy+y) & \text{if } 0 < x, y < 1 \\ 0 & \text{else.} \end{cases}$$

Calculate the marginals. Are the two variables independent?

**9.14** Calculate the marginal distributions! Are $X$ and $Y$ independent?

**HW** (a) (2 points) Let the joint probability density function of $X$ and $Y$ be

$$f(x,y) = \begin{cases} A \cdot (x^2y + y^2x) & \text{if } 0 < x, y < 1 \\ 0 & \text{else} \end{cases}$$

where $A$ is a positive constant that must be calculated first.

(b) Same question when $(X,Y)$ is uniform on the unit circle.

(c) $(X,Y)$ is a uniform distribution on $D = \{(x,y) \in \mathbb{R}^2 : \, 2x^2 + y^2/2 \leq 1\}$.

**9.15** A rectangle has sides with length 1 and $a$. We choose one point on each side of length 1, with independent uniform distributions. Let $X$ be the distance of these two points. What is the PDF of $X$?