P.D.F. TRANSFORMS:

IF $\mathbb{X}$ HAS P.D.F. $f(x)$ AND $\mathbb{Y} := \psi(\mathbb{X})$

THEN THE P.D.F. OF $\mathbb{Y}$ IS $\boxed{g(y) = \sum_{x:\,\psi(x)=y} f(x)\cdot \frac{1}{|\psi'(x)|}}$

$\boxed{9.3}$ (a) $\xi \sim \mathcal{N}(0,1)$, P.D.F. OF ξ^2? g

$$\psi(x) = x^2, \quad \psi'(x) = 2x, \quad \psi^{-1}(y) = \begin{cases} \emptyset & \text{IF } y<0 \\ \pm\sqrt{y} & \text{IF } y \geq 0 \end{cases}$$

THUS $g(y) = 0$ IF $y<0$ AND IF $y>0$ THEN

$$g(y) = \sum_{x=\pm\sqrt{y}} \varphi(x)\cdot\frac{1}{|2x|} = \varphi(\sqrt{y})\cdot\frac{1}{|2\sqrt{y}|} + \varphi(-\sqrt{y})\cdot\frac{1}{|2\sqrt{y}|}$$

$$\underset{\text{SYMMETRY}}{=} \frac{1}{\sqrt{y}}\cdot\varphi(\sqrt{y}) = \frac{1}{\sqrt{y}}\cdot\frac{1}{\sqrt{2\pi}}\cdot e^{-y/2}$$

SINCE $\varphi(x) = \frac{1}{\sqrt{2\pi}}\cdot e^{-x^2/2}$

ALTERNATIVE SOLUTION: $G(y) = \mathbb{P}(\mathbb{X} \leq y) = \mathbb{P}(\xi^2 \leq y) =$ $\boxed{\text{IF } y>0}$

$= \mathbb{P}(-\sqrt{y} \leq \xi \leq \sqrt{y}) = \Phi(\sqrt{y}) - \Phi(-\sqrt{y}) =$

$\Phi(\sqrt{y}) - (1-\Phi(\sqrt{y})) = 2\cdot\Phi(\sqrt{y}) - 1$

$g(y) = G'(y) = 2\cdot\Phi'(\sqrt{y})\cdot\frac{1}{2\cdot\sqrt{y}} = \varphi(\sqrt{y})\cdot\frac{1}{\sqrt{y}}$

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$\boxed{9.3}$ (d) $\xi \sim EXP(\lambda)$, $Y = e^{\xi}$ P.D.F.? $g(y)$

$$\psi(x) = e^{x}, \quad \psi'(x) = e^{x}, \quad \psi^{-1}(y) = \begin{cases} \emptyset & \text{IF } y \le 0 \\ \ln(y) & \text{IF } y > 0 \end{cases}$$

THUS $g(y) = 0$ IF $y \le 1$ (SINCE $\mathbb{P}(\xi \ge 0) = 1$, THUS $\mathbb{P}(e^{Y} \ge e^{0}) = 1$). NOW IF $y > 1$ THEN

$$g(y) = \sum_{x:\, \psi(x) = y} \lambda \cdot e^{-\lambda \cdot x} \cdot \frac{1}{|e^{x}|} = \lambda \cdot e^{-\lambda \cdot \ln(y)} \cdot \frac{1}{y} =$$

$$= \lambda \cdot y^{-\lambda} \cdot \frac{1}{y} = \lambda \cdot e^{-(\lambda+1)} \quad \text{IF } y > 1.$$

ALTERNATIVE SOLUTION:

$$G(y) = \mathbb{P}(Y \le y) = \mathbb{P}(e^{\xi} \le y) = \mathbb{P}(\xi \le \ln(y)) =$$

$$= \begin{cases} 0 & \text{IF } \ln(y) \le 0 \\ 1 - e^{-\lambda \cdot \ln(y)} & \text{IF } \ln(y) \ge 0 \end{cases}$$

$$= \begin{cases} 0 & \text{IF } y \le 1 \\ 1 - y^{-\lambda} & \text{IF } y \ge 1 \end{cases}, \quad \text{THUS}$$

$$g(y) = G'(y) = \begin{cases} 0 & \text{IF } y \le 1 \\ \lambda \cdot y^{-(\lambda+1)} & \text{IF } y > 1 \end{cases}$$

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$\boxed{9.2}$ ASSUME THAT F IS CONTINUOUS AND STRICTLY INCREASING

$$F(x) = \mathbb{P}(X \le x), \quad Y := F(X) \quad \boxed{F: \mathbb{R} \to (0,1)}$$

$$G(y) = \mathbb{P}(Y \le y) = \begin{cases} 0 & \text{IF } y \le 0 \\ \star & \text{IF } y \in (0,1) \\ 1 & \text{IF } y \ge 1 \end{cases}$$

$$\star = \mathbb{P}(F(X) \le y) = \mathbb{P}(X \le F^{-1}(y)) =$$

$$F(F^{-1}(y)) = y, \quad \text{THUS } \boxed{Y \sim \mathrm{UNI}[0,1]}$$

NOTE: $Y = F(X)$, THUS $\boxed{X = F^{-1}(Y)}$

THUS IF WE WANT TO GENERATE A RANDOM VARIABLE WITH C.D.F. F THEN IT IS ENOUGH TO GENERATE AN $Y \sim \mathrm{UNI}[0,1]$ AND THEN THE C.D.F. OF $F^{-1}(Y)$ WILL BE F

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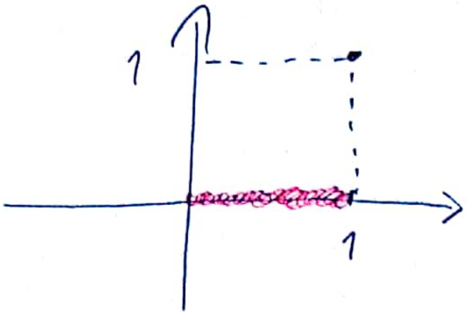
$\boxed{9.6}$ a)



$$\xi = \sqrt{1^2 + (1-\mathcal{X})^2}$$

ξ P.D.F. $g(y)$?

$$\psi(x) = \sqrt{1-(x-1)^2}, \quad \psi'(x) = \frac{2\cdot(x-1)}{2\cdot\sqrt{1-(x-1)^2}} \quad (\psi \searrow)$$

$$\psi^{-1}(y) = 1 - \sqrt{y^2-1} \quad \text{IF } 1 \le y \le \sqrt{2}$$

$$g(y) = \sum_{x:\psi(x)=y} f(x)\cdot\frac{1}{|\psi'(x)|} = 1\cdot\frac{1}{|\psi'(\psi^{-1}(y))|} =$$

$$= |(\psi^{-1})'(y)| = \frac{y}{\sqrt{y^2-1}} \quad \text{IF } 1 < y \le \sqrt{2}$$

(AND ZERO OTHERWISE)

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a) ALTERNATIVE SOLUTION

$$G(y) = \mathbb{P}(\xi \le y) = \begin{cases} 0 & \text{IF } y \le 1 \\ ☆ & \text{IF } y \in (1, \sqrt{2}) \\ 1 & \text{IF } y \ge \sqrt{2} \end{cases}$$

$$☆ = \mathbb{P}\left(\sqrt{1+(1-\mathcal{X})^2} \le y\right) = \mathbb{P}\left(1+(1-\mathcal{X})^2 \le y^2\right) =$$

$$= \mathbb{P}\left(1-\mathcal{X} \le \sqrt{y^2-1}\right) = \mathbb{P}\left(1-\sqrt{y^2-1} \le \mathcal{X}\right) =$$

$$= 1-\left(1-\sqrt{y^2-1}\right) = \sqrt{y^2-1}, \text{ THUS}$$

SINCE $\mathcal{X} \sim \text{UNI}[0,1]$

$$g(y) = G'(y) = \begin{cases} 0 & \text{IF } y \le 1 \\ \frac{y}{\sqrt{y^2-1}} & \text{IF } y \in (1, \sqrt{2}) \\ 0 & \text{IF } y \ge \sqrt{2} \end{cases}$$

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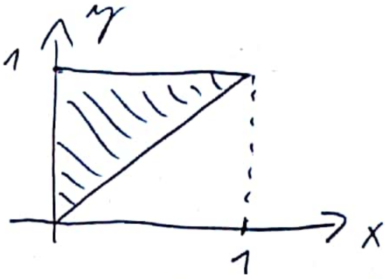
$\boxed{9.7}$ $X \sim UNI[0,1]$, $Y \sim UNI[0,1]$

X AND Y ARE INDEPENDENT (ASSUME THAT $\ell = 1$)

THUS (X, Y) IS UNIFORMLY DISTRIBUTED ON THE UNIT SQUARE $[0,1]^2$. LET

$X^* := X \wedge Y$, $Y^* := X \vee Y$, THEN

(X^*, Y^*) IS UNIFORMLY DISTRIBUTED ON THIS TRIANGLE:



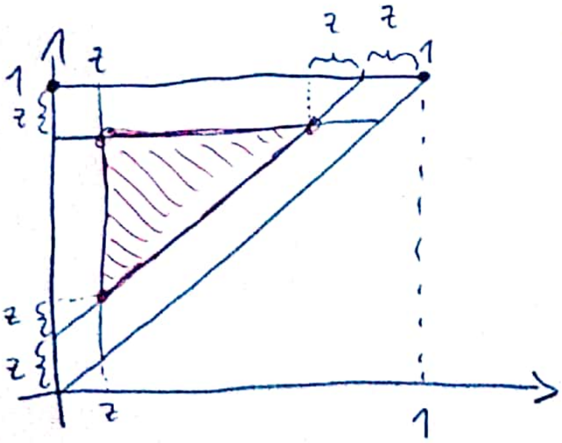
LENGTHS OF THE THREE PIECES:

$X^*, \ Y^* - X^*, \ 1 - Y^*$

LENGTH OF SHORTEST: $Z_1 := X^* \wedge (Y^* - X^*) \wedge (1 - Y^*)$

NOTE: $\mathbb{P}(0 \le Z_1 \le \frac{1}{3}) = 1$ AND IF $z \in [0, \frac{1}{3}]$ THEN

$$\mathbb{P}(Z_1 \ge z) = \frac{\text{RED AREA}}{\text{TOTAL AREA}} = \frac{\frac{1}{2} \cdot (1-3z)^2}{1/2}$$



$$\mathbb{E}(Z_1) = \int_0^\infty \mathbb{P}(Z_1 \ge z)\, dz =$$

$$= \int_0^{1/3} (1-3z)^3 \, dz = \frac{1}{9}$$

AND IF THE STICK HAS LENGTH ℓ THEN $\mathbb{E}(\text{SHORTEST}) = \ell/9$ BY THE LINEARITY OF EXPECTATION

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$\boxed{9.8}$ b) $i \in \{1,\dots,6\}$, $j \in \{1,\dots,6\}$

$$p(i,j) = \mathbb{P}(\text{FIRST} = i,\ \text{MAX} = j)$$

IF $i > j$ THEN $p(i,j) = 0$

IF $i < j$ THEN

$$p(i,j) = \mathbb{P}(\text{FIRST} = i) \cdot \mathbb{P}(\text{SECOND} = j) = \frac{1}{6} \cdot \frac{1}{6}$$

IF $i = j$ THEN

$$p(i,j) = \mathbb{P}(\text{FIRST} = i) \cdot \mathbb{P}(\text{SECOND} \le j) = \frac{1}{6} \cdot \frac{i}{6}$$

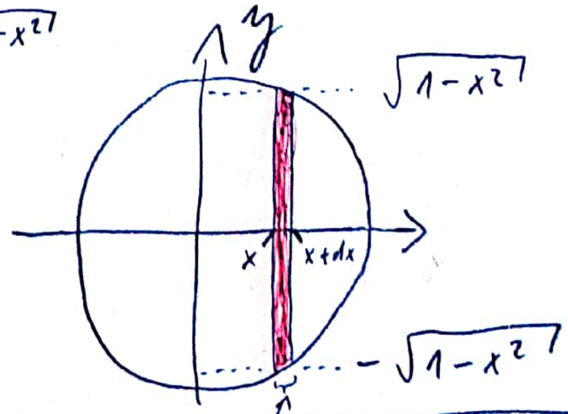
$\boxed{9.11}$ JOINT P.D.F.: $f(x,y) = \frac{1}{\pi} \cdot \mathbb{1}[x^2+y^2 \le 1]$

π IS THE AREA OF UNIT DISC, SO $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \frac{1}{\pi}\int_{-\infty}^{\infty} \mathbb{1}[x^2+y^2 \le 1]\,dy =$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \mathbb{1}[|y| \le \sqrt{1-x^2}]\,dy = \frac{1}{\pi}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy = \frac{2\cdot\sqrt{1-x^2}}{\pi},$$

$$f_X(x)\,dx = \mathbb{P}(X \in [x, x+dx]) =$$



$\boxed{\text{WIDTH} = dx}$

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$\boxed{9.12}$ IF (X, Y) ARE JOINTLY ABSOLUTELY CONTINUOUS

THEN $\boxed{X \text{ AND } Y \text{ ARE INDEP.}}$ IF AND ONLY IF

$\boxed{f(x,y) = f_X(x) \cdot f_Y(y)}$: JOINT P.D.F. IS OF PRODUCT FORM.

a) $$f(x,y) = \underbrace{x \cdot e^{-x} \cdot \mathbb{1}[x>0]}_{\text{FUNCTION OF } x} \cdot \underbrace{e^{-y} \cdot \mathbb{1}[y>0]}_{\text{FUNCTION OF } y}$$

$$f_X(x) = x \cdot e^{-x} \cdot \mathbb{1}[x>0], \quad f_Y(y) = e^{-y} \cdot \mathbb{1}[y>0]$$

THUS X AND Y ARE INDEPENDENT.

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b) $f(x,y) = 2 \cdot \mathbb{1}[0<x<y<1]$ IS NOT OF PRODUCT FORM.

$$f_X(x) = \int_x^1 2\,dy = 2 \cdot (1-x) \text{ IF } 0<x<1 \quad \left(\text{OTHERWISE ZERO}\right)$$

$$f_Y(y) = \int_0^y 2\,dx = 2 \cdot y \text{ IF } 0<y<1 \quad \left(\text{OTHERWISE ZERO}\right)$$

$f_X(x) \cdot f_Y(y) \neq f(x,y)$ THUS X AND Y ARE NOT INDEPENDENT.

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