Probability 1 – Exercises

Tutorial no. 10 | 7th Nov 2024

9.10 Let X and Y be random variables with joint probability density function

$$f(x,y) = \begin{cases} \frac{4}{5}(x + xy + y), & \text{if } 0 < x, y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Calculate the marginals. Are the two variables independent?

10.1 *(Buffon's needle problem)* Suppose we have a floor made of parallel strips of wood, each the same width t, and we drop a needle of length l onto the floor. Suppose that the length of the needle is shorter than the width of the strips. What is the probability that the needle will lie across a line between two strips?

10.2 Let $X_1, X_2, \dots \sim \mathrm{Exp}(\lambda)$ be independent exponential random variables.

(a) What is the distribution of $Z = X_1 + X_2$?

(b) What is the probability that $X_1 + X_2 + X_3 > t$?

(c) What is the expectation and variance of $X_1 + X_2 + X_3$?

HW 10.3 (3 points) $X_1 \sim \mathrm{Exp}(\lambda_1)$ and $X_2 \sim \mathrm{Exp}(\lambda_2)$ are two independent exponential random variables. Assume $0 < \lambda_1 < \lambda_2$. What is the distribution of $Z = X_1 + X_2$?

10.4 Let the joint PDF of X and Y be

$$f(x,y) = \begin{cases} \frac{1}{y}\exp(-(y + x/y)) & \text{, if } x > 0, y > 0 \\ 0 & \text{, else.} \end{cases}$$

(a) What is the marginal distribution of Y? How much is $\mathbb{E}(Y)$?

(b) Note that calculating the marginal distribution of X explicitely is problematic. Without doing that, calculate $\mathbb{E}(X)$.

(c) What is the conditional density of X, given $Y = y$?

HW 10.5 (3 points) Let the joint density of (X, Y) be $f(x,y) = 24xy\mathbf{1}_{0<x,\, 0<y,\, x+y<1}$.

(a) What is the marginal density of X?

(b) What is the conditional density of Y given $X = x$?

10.6 N people arrive for a business dinner. Whenever someone arrives, they look around, and if they see one of their friends, they sit down to one of their tables, otherwise, they sit to a new, empty table. Each person is friends with any other person with p probability, independently from each other. Calculate the expected value of the number of occupied tables.

10.7 10 married couples are seated around a round table, and they all sit down randomly. Let X be the number of couples that manage to sit down next to each other. What is the expected value and variance of X?

10.8 (a) I go to work each day via train and then bus. First, I get on a train, which arrives at the train station at 7:30am, according to schedule. I have 2 minutes to walk to the bus station, where our bus leaves at 7:37am, according to schedule. However, the actual times when the train arrives and the bus leaves are random. More precisely, the train arrives according to a Normal Distribution with a mean of 7:30am and standard deviation of 3, while the time when the bus leaves is normal with mean 7.37am and standard deviation of 4. What is the probability that I only miss the bus (at most) once in a given week (which consists of 5 work days)?

(b) In the afternoon, my bus arrives back to the station at 17:40 and my train leaves at 17:49 according to schedule, with the same standard deviations as above. What is the probability that in the 220 workdays of the year, I will miss the afternoon connection more often than my morning connection?

10.9 On a cold foggy evening, starting at 7 pm, I'm waiting for my girlfriend's bus to arrive. These buses come according to a Poisson process with intensity 1 bus per 10 minutes. Already, the second bus arrives at 7:22 pm, but my girlfriend is not on the bus. At this point, she suddenly appears from the fog, saying that she arrived with the previous bus, we just didn't see each other. Not remembering when the first bus actually came, what is the distribution of the time we spent standing next to each other in the fog?

10.10 Let $X \sim \mathcal{N}(0,1), Y \sim \mathcal{N}(0,3)$ be independent from each other. Let M be a randomly choosen point on the plane with coordinates (X, Y). What is the probability that

(a) $M \in \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 2, |y| \leq 1\}$;

(b) $M \in \{(x,y) \in \mathbb{R}^2 : x + y < 3\}$;

HW (c) (2 points) $M \in \{(x,y) \in \mathbb{R}^2 : 1 \leq \sqrt{3}|x| + |y| \leq 3\}$ (Hint: Rotate the domain);

HW (d) (2 points) $M \in \{(x,y) \in \mathbb{R}^2 : 1 \leq 3x^2 + y^2 \leq 3\}$ (Hint: Polar coordinates);

♣ (e) (3 points) $M \in \{(x,y) \in \mathbb{R}^2 : x + y \leq 0, -2 \leq y, |x| \leq 1\}$?

10.11 We break an L long stick into two parts, by breaking it at a uniformly chosen point.

(a) Let X be the squared sum of the length of the two parts. What is the PDF of X?

(b) Let Y be the product of the length of the two parts. What is the PDF of Y?